PROBLEM OF THE WEEK

Solution of Problem No. 13 (Fall 2005 Series)

Problem: The medians of the triangle T divide it into 6 smaller triangles. Show that their centroids lie on an ellipse in the interior of T, centered at the centroid of T.

Solution

This problem is essentially the same as Problem 13, Spring 2002. The Panel apologizes for the oversight.

Let ABC' be an equilateral triangle with the same "base" AB. Then there exists a linear invertible transformation L that maps ABC into ABC'. To construct it, one can assume that A is the origin, then define L as the unique linear transformation in $\mathbb{R}^2$ that maps B into B, and C into C'. It is invertible, because it maps a pair of linearly independent vectors into a pair of linearly independent vectors.

Since linear transformations preserve ratios, centroids, map ellipses into ellipses by preserving their centers, the problem is then reduced to an one for an equilateral triangle. In that case however, all the six small triangles are congruent and it is easy to show that they stay at the same distance to the centroid of T. A direct calculation shows that this distance is smaller than the distance from o to each side. Therefore, the six centroids lie on a circle that is in the interior of T. Under the inverse transformation L^{-1}, this circle is transformed into an ellipse, still lying inside the triangle.

At least partially solved by:

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Update on POW #12: Solved also by Steven Landy.