PROBLEM OF THE WEEK

Solution of Problem No. 14 (Fall 2005 Series)

Problem: Given a triangle ABC, find a triangle $A_1B_1C_1$, so that

(1) $A_1 \in BC$, $B_1 \in CA$, $C_1 \in AB$;

(2) the centroids of $\triangle ABC$ and $\triangle A_1B_1C_1$ coincide; and

subject to (1) and (2), $\triangle A_1B_1C_1$ has minimal area.

Solution (by Georges Ghosn (Quebec)

There are 3 real numbers α, β and γ in (0,1) which verify:

$$\vec{BA_1} = \alpha \vec{BC} \quad \vec{CB_1} = \beta \vec{CA} \quad \text{and} \quad \vec{AC_1} = \gamma \vec{AB}.$$

The centroids of $\triangle ABC$ and $\triangle A_1B_1C_1$ coincide implies the existence of a point G such that $\vec{GA} + \vec{GB} + \vec{GC} = \vec{O}$ and $\vec{GC_1} + \vec{GA_1} + \vec{GB_1} = \vec{O}$. Therefore by subtracking these relations we get: $\vec{AC_1} + \vec{BA_1} + \vec{CB_1} = \vec{O} \Leftrightarrow \gamma\vec{AB} + \alpha\vec{BC} + \beta\vec{CA} = 0 \Leftrightarrow (\gamma - \beta)\vec{AB} + (\alpha - \beta)\vec{BC} = \vec{O} \Leftrightarrow \alpha = \beta = \gamma$ since $\vec{AB}$ and $\vec{BC}$ are non–colinear vectors. Therefore the above logical equivalences show that the conditions $\alpha = \beta = \gamma$ is a necessary and sufficient conditions for the centroids of $\triangle ABC$ and $\triangle A_1B_1C_1$ to coincide. On the other hand, from the area of a triangle formula: area of $\triangle ABC = \frac{1}{2}bc\ \sin(A) = \frac{1}{2}ac\ \sin(B) = \frac{1}{2}ab\ \sin(C)$, we deduce: area of $\triangle AB_1C_1$ = area of $\triangle BA_1C_1$ = area of $\triangle CA_1B_1 = \alpha(1-\alpha)$(area of $\triangle ABC$). Therefore area of $\triangle A_1B_1C_1$ is minimal if and only if $\alpha(1-\alpha)$ is maximal. Therefore $\alpha = \frac{1}{2}$ and A_1, B_1 and C_1 are the midpoints of BC, CA and AB respectively.

At least partially solved by:

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