PROBLEM OF THE WEEK

Solution of Problem No. 2 (Fall 2005 Series)

Problem: Find a number $k > 0$ such that any sequence of real numbers a_n, $n = 0, 1, 2...$, satisfying $a_{n+2} = ka_{n+1} - a_n$ for all n must also satisfy $a_{n+8} = a_n$ for all n.

Solution (by the Panel)

The characteristic equation is

$$\lambda^2 - k\lambda + 1 = 0$$

with roots

$$r_{1,2} = \frac{1}{2}(k \pm \sqrt{k^2 - 4})$$

that can be real and distinct ($k > 2$), real and repeated ($k = 2$), and complex conjugate ($0 < k < 2$).

If $k = 2$, then $a_n = c_1 + c_2 n$ with two constants $c_1 c_2$, and a_n is not periodic if $c_2 \neq 0$ (so, it is not true that any sequence solving the recursive equation is periodic with period 8).

If $k \neq 2$, then $a_n = c_1 r_1^n + c_2 r_2^n$. Since $a_{n+8} = a_n$ for any n and any sequence, i.e., for any two constants c_1, c_2, we get

$$r_1^{n+8} = r_1^n, \quad r_2^{n+8} = r_2^n,$$

therefore, r_1, r_2 solve also $x^8 - 1 = 0$. The roots of the latter are given by $x_\ell = e^{2\pi i \ell/8}$, $\ell = 0, \ldots, 7$, where $i = \sqrt{-1}$. It is fairly easy to see that only $k = \sqrt{2}$ gives r_1, r_2 among those 8 roots.

On the other hand, if $k = \sqrt{2}$, then

$$r_{1,2} = \frac{\sqrt{2}}{2}(1 \pm i),$$

$r_1^8 = r_2^8 = 1$, therefore

$$a_{n+8} = a_n$$

for any choice of c_1, c_2.

Many solvers found the right value of k, but very few proved that this was the only one, and/or that for $k = \sqrt{2}$, any sequence satisfies $a_{n+8} = a_n$.

Solved by:

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Problem No. 1 was also solved by Kevin Laster, whose name was omitted last week.