PROBLEM OF THE WEEK
Solution of Problem No. 4 (Fall 2005 Series)

Problem: Fix positive real numbers ρ and γ with $\gamma < \pi$. For triangles $\triangle$ with inradius ρ and one angle γ (radians), determine the smallest possible radius of the circumcircle of $\triangle$.

Show that f must take on odd values at an even number of the 8 vertices.

Solution (by the Panel)

Let a, b, c be the sides of $\triangle$, and let α, β, γ be the corresponding angles. By the Law of sines,

$$R = \frac{c}{2\sin\gamma}.$$

On the other hand, it is easy to see that

$$c = \rho\left(\cot\frac{\alpha}{2} + \cot\frac{\beta}{2}\right).$$

Since c and γ are fixed, we need to minimize

$$f(\alpha, \beta) = \cot\frac{\alpha}{2} + \cot\frac{\beta}{2}$$

under the constraint $\frac{\alpha}{2} + \frac{\beta}{2} = \frac{\pi}{2} - \frac{\gamma}{2}$, $0 \le \alpha$, $0 \le \beta$. The function $g(x) = \cot x$ is strictly convex on the interval $(0, \pi/2)$ because

$$g''(x) = 2\cot x\,(1 + \cot^2 x) > 0.$$

Therefore,

$$g\left(\frac{x+y}{2}\right) \le \frac{g(x) + g(y)}{2}, \quad x, y \in \left(0, \frac{\pi}{2}\right).$$

and there is equality only if $x = y$.

Set $x = \alpha/2$, $y = \beta/2$ above to get

$$\begin{aligned} \cot\frac{\alpha}{2} + \cot\frac{\beta}{2} &\ge 2\cot\frac{\alpha+\beta}{4} \\ &= 2\cot\frac{\pi-\gamma}{4} \end{aligned}$$

with equelity if and only if $\alpha = \beta = (\pi - \gamma)/2$.

Therefore the minimal value for R is

$$R_{\min} = \rho \cot \frac{\pi - \gamma}{4} \Big/ \sin \gamma,$$

and it is attained when $\alpha = \beta = \frac{\pi-\gamma}{2}$.

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