PROBLEM OF THE WEEK

Solution of Problem No. 5 (Fall 2005 Series)

Problem: Let f be a real–valued function with continuous non–negative derivative. Assume that $f(0) = 0$, $f(1) = 1$, and let ℓ be the length of the graph of f on the interval [0,1].

Prove that

$$\sqrt{2} \leq \ell < 2.$$

Solution (by the Panel)

The following inequalities are well known and easy to verify

$$\frac{\sqrt{2}}{2}(a+b) \leq \sqrt{a^2+b^2} \leq a+b,$$

if $a \geq 0$, $b \geq 0$. The second one turns into equality if and only if $ab = 0$.

Now,

$$\ell = \int_0^1 \sqrt{1+(f'(x))^2}\, dx.$$

Therefore, since $f'(x) \geq 0$,

$$\sqrt{2} = \frac{\sqrt{2}}{2}\int_0^1 (1+f'(x))\, dx \leq \ell \leq \int_0^1 (1+f'(x))\, dx = 2,$$

$$\text{i.e.,} \qquad \sqrt{2} \leq \ell \leq 2.$$

If $\ell = 2$, then we must have

$$\sqrt{1+(f'(x))^2} = 1 + f'(x), \quad \forall x \in [0,1].$$

This implies $f'(x) = 0, \quad \forall x \in [0,1]$, thus $f = \mathit{const}$. The latter contradicts the conditions $f(0) = 0, f(1) = 1$.

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