PROBLEM OF THE WEEK

Solution of Problem No. 6 (Fall 2005 Series)

Problem: Let ϕ be the Euler function defined by $\phi(1) = 1$, and for any integer $n > 1$, $\phi(n)$ is the number of positive integers $\leq n$ and relatively prime to n. Prove that for all real $x \neq \pm 1$.

$$\sum_{m=0}^{\infty} (-1)^m \phi(2m+1) \frac{x^{2m+1}}{1+x^{4m+2}} = \frac{|x-x^3|}{(1+x^2)^2}.$$

Solution (by Georges Ghosn, Quebec)

We suppose first that $|x| < 1$, in this case the series $\sum (-1)^m \Phi(2m+1) \frac{x^{2m+1}}{1+x^{4m+2}}$ is absolutely convergent. Indeed, $\left| (-1)^m \Phi(2m+1) \frac{x^{2m+1}}{1+x^{4m+2}} \right| \leq (2m+1)|x|^{2m+1}$ and it is easy to show that $\sum (2m+1)|x|^{2m+1}$ converges over $]-1, 1[$. On the other hand, we have for $|x| < 1$, $\frac{1}{1+x} = \sum_{n=0}^{+\infty} (-1)^n x^n$, therefore $\frac{x}{1+x^2} = \sum_{n=0}^{+\infty} (-1)^n x^{2n+1}$ and for $m \geq 0$,

$$\frac{x^{2m+1}}{1+x^{4m+2}} = \frac{x^{2m+1}}{1+(x^{2m+1})^2} = \sum_{n=0}^{+\infty} (-1)^n x^{(2m+1)(2n+1)} \quad (|x|^{2m+1} < 1).$$

Therefore:

$$\sum_{n=0}^{+\infty} (-1)^m \Phi(2m+1) \frac{x^{2m+1}}{1+x^{4m+2}} = \sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} (-1)^{m+n} \Phi(2m+1) x^{(2m+1)(2n+1)}.$$

In order to reorder terms of this double series, we must prove that it is absolutely convergent. Indeed this double series is absolutely convegent over any compact $[-\alpha, \alpha]$
$0 \leq \alpha < 1$ because:

$$\left| (-1)^m \Phi(2m+1) x^{(2m+1)(2n+1)} \right| \leq (2m+1) \alpha^{(2m+1)(2n+1)}$$

$$\text{and} \quad \sum_{m=0}^{M} \sum_{n=0}^{+\infty} (2m+1) \alpha^{(2m+1)(2n+1)} = \sum_{m=0}^{M} (2m+1) \frac{\alpha^{2m+1}}{1-\alpha^{4m+2}}$$

$$\text{and} \quad \forall m \geq 0 \quad (2m+1) \frac{\alpha^{2m+1}}{1-\alpha^{4m+2}} \leq (2m+1) \frac{\alpha^{2m+1}}{1-\alpha^2}$$

$$\text{and} \quad \sum \frac{(2m+1)\alpha^{2m+1}}{1-\alpha^2} \quad \text{converge for} \quad 0 \leq \alpha < 1.$$

Therefore

$$\sum_{m=0}^{+\infty}(-1)^m\Phi(2m+1)\frac{x^{2m+1}}{1+x^{4m+2}} = \sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}(-1)^{m+n}\Phi(2n+1)x^{(2m+1)(2n+1)} = \sum_{p=0}^{+\infty}a_p x^{2p+1}$$

where a_p is a summation extended over all terms $(-1)^{m+n}\Phi(2m+1)$ where $(2m+1)(2n+1) = 2p+1$. But $(2m+1)(2n+1) = 2p+1 \Leftrightarrow p = 2mn+m+n$ $\Rightarrow (-1)^{m+n} = (-1)^p$.

Therefore $a_p = (-1)^p \sum_{\substack{d|(2p+1)\\ d>0}} \Phi(d) = (-1)^p(2p+1)$. (Euler Function Properties)

Finally

$$\begin{aligned}
\sum_{p=0}^{+\infty}(-1)^p(2p+1)x^{2p+1} &= x\sum_{p=0}^{+\infty}(-1)^p x^{2p} + x^2\sum_{p=1}^{+\infty}(-1)^p 2px^{2p-1}\\
&= x\sum_{p=0}^{+\infty}(-1)^p(x^2)^p + x^2\left(\sum_{p=0}^{+\infty}(-1)^p x^{2p}\right)' = \frac{x}{1+x^2} + x^2\left(\frac{1}{1+x^2}\right)'\\
&= \frac{x}{1+x^2} - \frac{2x^3}{(1+x^2)^2} = \frac{x-x^3}{(1+x^2)^2}.
\end{aligned}$$

Now for $|x|>1$, we have: $\dfrac{x^{2m+1}}{1+x^{4m+2}} = \dfrac{(\frac{1}{x})^{2m+1}}{1+\left(\frac{1}{x}\right)^{4m+2}}$ and $\left|\dfrac{1}{x}\right| < 1$. Therefore

$$\begin{aligned}
\sum_{m=0}^{+\infty}(-1)^m\phi(2m+1)\frac{x^{2m+1}}{1+x^{4m+2}} &= \sum_{m=0}^{+\infty}(-1)^m\Phi(2m+1)\frac{(\frac{1}{x})^{2m+1}}{1+(\frac{1}{x})^{4m+2}}\\
&= \frac{\frac{1}{x}-(\frac{1}{x})^3}{\left(1+(\frac{1}{x})^2\right)^2} = \frac{x^3-x}{(1+x^2)^2}.
\end{aligned}$$

This completes the proof.

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