PROBLEM OF THE WEEK
Solution of Problem No. 7 (Fall 2005 Series)

Problem: Show that, for all positive integers m, n,

$$\frac{1.5.9....(4n-3).3.7.11....(4m-1).2^{m+n-1}}{(m+n)!}$$

is an integer.

Solution I (by Wing-kai Hon, CS, post–doc)

Let N and D denote the numerator and denominator of the above term, respectively. Note that

$$N = |(-(4m-1)) \times \cdots \times (-11) \times (-7) \times (-3) \times 1 \times 5 \times 9 \times \cdots \times (4n-3)| \times 2^{m+n-1}$$

where the first $m+n$ numbers in the $\cdots$ sign forms an arithmetic progression.

Let $N = \Pi p_i^{n_i}$ and $D = \Pi p_i^{d_i}$ denote the unique prime factorization of N and D. If follows that for $p_i = 2$, $n_i = m+n-1$ and $d_i = \lfloor \frac{m+n}{2} \rfloor + \lfloor \frac{m+n}{4} \rfloor + \lfloor \frac{m+n}{8} \rfloor + \ldots$ and for any p_i with $2 < p_i \leq m+n$,

$$n_i \geq \lfloor \frac{m+n}{p_i} \rfloor + \lfloor \frac{m+n}{p_i^2} \rfloor + \lfloor \frac{m+n}{p_i^3} \rfloor + \ldots$$
$$d_i = \lfloor \frac{m+n}{p_i} \rfloor + \lfloor \frac{m+n}{p_i^2} \rfloor + \lfloor \frac{m+n}{p_i^3} \rfloor + \ldots.$$

In other words,

$$n_i \geq d_i \qquad \forall p_i \leq m+n. \quad \text{(i.e.,} \quad n_i \geq d_i \quad \forall p_i/m+n)$$

Thus, $\frac{N}{D}$ is an integer.

Solution II (by Georges Ghosn, Quebec, edited by the Panel. We present a sketch of the proof only.)

We pose $A(n,m) = \frac{1.5.9....(4n-3).3.7.11....(4m-1).2^{m+n-1}}{(m+n)!}$ $(m \geq 1, m \geq 1)$. We define $R(k)$, row of rank $k (k \geq 2)$, as the set of all $A(n,m)$ such that $m+n = k, m \geq 1, n \geq 1$.

We will proceed as follow:

1) show that $R(2)$ and $R(3)$ are sets of integers.

2) show that for any $k \geq 2$, there is at least one integer belonging to $R(k)$, in particular, if $k = 2n$, $A(n, n)$ is an integer and if $k = 2n + 1$, $A(n + 1, n)$ is an integer.
3) for all m, n positive ≥ 2, we show that

$$A(n, m) + A(n + 1, m - 1) = 8A(n, m - 1)$$

and by deduction

$$A(n, m) + A(n - 1, m + 1) = 8A(n - 1, m).$$

Finally, by Mathematical induction on k we will show that $R(k)$ is a set of integers.

Also solved by:

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