PROBLEM OF THE WEEK

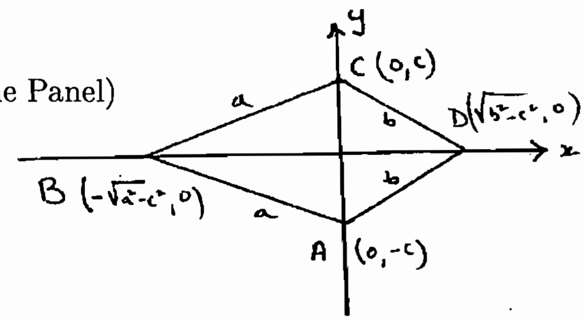
Solution of Problem No. 1 (Fall 2006 Series)

Problem: Let $a > b > 0$ be fixed numbers. Let Q be a convex planar quadrilateral with consecutive vertices A, B, C, D such that

$$|AB| = |BC| = a, \qquad |AD| = |DC| = b.$$

Determine the extreme values of the distance between the center of mass of the vertices of Q and the center of mass of Q as a plane region.

Solution (by Georges Ghosn, Quebec; edited by the Panel)



Observe that BD is the perpendicular bisector of AC. Therefore in the coordinate system using BD as x–axis and AC as y–axis we have:

$$A(0,-c) \quad B(-\sqrt{a^2-c^2},0) \quad C(0,c) \quad D(\sqrt{b^2-c^2},0), \quad 0 < c \le b.$$

The center of mass of the vertices of Q is : $I\left(\dfrac{\sqrt{b^2-c^2}-\sqrt{a^2-c^2}}{4},0\right)$.

The center of mass of the plane region Q is : $X_G = \dfrac{\int\int_Q xdxdy}{\text{area of } Q}, \quad Y_G = 0.$

But

$$\int\int_Q xdxdy = \int_{-\sqrt{a^2-c^2}}^{0} xdx \int_{-c\left(1+\frac{x}{\sqrt{a^2-c^2}}\right)}^{c\left(1+\frac{x}{\sqrt{a^2-c^2}}\right)} dy + \int_0^{\sqrt{b^2-c^2}} xdx \int_{-c\left(1-\frac{x}{\sqrt{b^2-c^2}}\right)}^{c\left(1-\frac{x}{\sqrt{b^2-c^2}}\right)} dy$$
$$= \frac{-c(a^2-c^2)+c(b^2-c^2)}{3}$$

and Area of $Q = c\sqrt{a^2-c^2} + c\sqrt{b^2-c^2}$.

Therefore $X_G = \dfrac{c(\sqrt{b^2-c^2}+\sqrt{a^2-c^2})(\sqrt{b^2-c^2}-\sqrt{a^2-c^2})}{3c(\sqrt{a^2-c^2}+\sqrt{b^2-c^2})} = \dfrac{\sqrt{(b^2-c^2)}-\sqrt{(a^2-c^2)}}{3}.$

Finally, the distance is $|IG| = f(C) = \dfrac{\sqrt{(a^2-c^2)} - \sqrt{(b^2-c^2)}}{12}$, $0 < c \leq b$. Next, f is an ncreasing continuous function on $[0, b]$ since

$$f'(C) = \frac{c}{12}\left(\frac{\sqrt{(a^2-c^2)} - \sqrt{(b^2-c^2)}}{\sqrt{(a^2-c^2)(b^2-c^2)}}\right) > 0$$

on $(0, b)$. Therefore the extreme values are $\dfrac{\sqrt{(a^2-b^2)}}{12}$ (for $c = b$) and $\dfrac{a-b}{12}$ (for $c = 0$). The last one is not reached since $c \neq 0$.

Also, at least partially solved by:

Undergraduates: Alan Bernstein (Sr. ECE), Ramul Kumar (So. E)

Graduates: Tom Engelsman (ECE), Jim Vaught (ECE)

Others: Hoan Duong (San Antonio College), Steven Landy (IUPUI Physics staff)