PROBLEM OF THE WEEK

Solution of Problem No. 11 (Fall 2006 Series)

Problem:

Identical beads are distributed among the vertices of a regular octagon in such a way that the center of mass of the distribution is at the center of the octagon.

(a) Show that the number of beads at any vertex is the same as that at the diametrically opposite vertex.

(b) Is the conclusion of (a) true if the octagon as replaced by a hexagon?

Solution (by the Panel)

(a) We can assume that the center is $(0,0)$ and the vertices are $(\pm 1, 0), (0, \pm 1)$, $\left(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2}\right)$, ordered so that the first one is $(1,0)$, and each subsequent one is obtained by rotating by $\pi/4$ in counter–clockwise direction. Let m_i be the number of beads at the i–th vertex. Then

$$m_1(1,0) + m_2(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}) + m_3(0,1) + m_4\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$
$$+ m_5(-1,0) + m_6\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) + m_7(0,-1) + m_8\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) = 0.$$

Comparing the first coordinates, we get

$$m_1 - m_5 + \frac{\sqrt{2}}{2}(m_2 - m_4 - m_6 + m_8) = 0.$$

Since $\frac{\sqrt{2}}{2}$ is an irrational number, we must have $m_1 = m_5$ (and $m_2 - m_4 - m_6 + m_8 = 0$). We can now rotate the coordinate system by $\pi/4$ to get $m_2 = m_6$, etc.

(b) No. Place 1 bead at each one of the vertices 1, 3, 5 and 2 beads at each one of the vertices 2, 4, 6.

At least partially solved by:

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