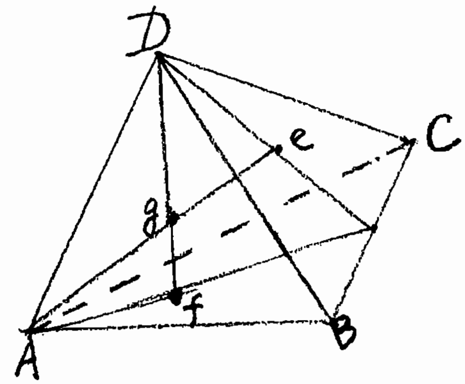
PROBLEM OF THE WEEK

Solution of Problem No. 12 (Fall 2006 Series)

Problem:

Prove that the altitudes of a non–degenerate tetrahedron meet in a point if and only if each pair of opposite edges is orthogonal.



Solution (by Steven Landy, edited by the Panel)

Consider tetrahedron $ABCD$ with altitudes Ae and Df. If Ae and Df intersect in g, then $ADefg$ are coplanar.

$$Ae \perp \text{ plane } (BCD) \Rightarrow Ae \perp BC$$
$$Df \perp \text{ plane } (ABC) \Rightarrow Df \perp BC.$$

Therefore, plane $(ADefg) \perp BC \Rightarrow AD \perp BC$. Similarly, $AB \perp DC, AC \perp BD$.

Now suppose $AD \perp BC$. Since $Ae \perp$ plane (BCD), $Ae \perp BC$. Thus, $BC \perp$ plane (ADe). Similarly, $BC \perp$ plane (ADf).

Therefore, Ae and Df are coplanar, so they intersect. Similarly all altitudes meet pairwise. Since no three of them are coplanar, they must meet in one point.

At least partially solved by:

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