PROBLEM OF THE WEEK

Solution of Problem No. 13 (Fall 2006 Series)

Problem:

Let $a_n > 0$ be a decreasing sequence, such that $\sum\limits_{n=1}^{\infty} a_n$ converges. Let $b_n > 0$ be a bounded sequence. Prove that

$$\sum_{n=1}^{\infty} (b_1 + \cdots + b_n)(a_n - a_{n-1})$$

converges.

Solution (by Georges Ghosn, QUEBEC, edited by the Panel)

Consider the sequence $u_n = (b_1 + \cdots + b_n)(a_{n-1} - a_n)$ for $n \geq 2$ and its partial sum $S_N = \sum\limits_{n=2}^{N} u_n$. We can easily show that $0 \leq u_n \leq Mn(a_{n-1} - a_n)$, where $M > 0$ is an upper bound of $\{b_n\}$. Therefore,

$$0 \leq S_N \leq M \sum_{n=2}^{N} (na_{n-1} - na_n) = M\left(\sum_{n=2}^{N} na_{n-1} - \sum_{n=2}^{N} na_n \right).$$

By shifting the first summation index, we get:

$$0 \leq S_N \leq M\left(\sum_{n=1}^{N-1} (n+1)a_n - \sum_{n=2}^{N} na_n \right) = M\left(2a_1 + \sum_{n=2}^{N-1} a_n - Na_N \right) \leq M\left(a_1 + \sum_{n=1}^{N-1} a_n \right).$$

Therefore, $0 \leq S_N \leq M\left(a_1 + \sum\limits_{n=1}^{\infty} a_n \right)$. Hence S_N converges since it is monotone and bounded sequence. Finally, $\sum\limits_{n=2}^{\infty} (b_1 + \cdots + b_n)(a_n - a_{n-1})$ converges and is equal to $-\lim\limits_{N\to\infty} S_N$, or $-\sum\limits_{n=2}^{\infty} u_n$.

——There are no other correct solutions for this problem.——