PROBLEM OF THE WEEK

Solution of Problem No. 14 (Fall 2006 Series)

Problem:

Find all positive integers n such that the decimal expansion of $n!$ ends with 2006 zeros but not with 2007 zeros. Use of computers or calculators is not allowed.

Solution (by Georges Ghosn, Quebec, edited by the Panel)

The exponent of a prime p in the prime factorization of $n!$ is equal to $\displaystyle\sum_{i=1}^{\infty}\left[\frac{n}{p^i}\right]$ which is a finite summation because $\left[\dfrac{n}{p^i}\right] = 0$ for $i > \log_p n$. Here $[x]$ is the integer part of x.

Therefore, $n! = 2^a \cdot 5^b \cdots$ with $a = \displaystyle\sum_{i=1}^{\infty}\left[\frac{n}{2^i}\right]$ and $b = \displaystyle\sum_{i=1}^{\infty}\left[\frac{n}{5^i}\right]$.

But for $n \geq 2$, we have $b < a$, therefore in order to have the decimal expansion of $n!$ ends with exactly 2006 zeros, b must be equal to 2006.

But $b = 2006 = \displaystyle\sum_{i=1}^{\infty}\left[\frac{n}{5^i}\right] < \sum_{i=1}^{\infty}\frac{n}{5^i} = \frac{n}{5}\sum_{i=0}^{\infty}\left(\frac{1}{5}\right)^i = \frac{n}{5} \times \frac{1}{1-\frac{1}{5}} = \frac{n}{4}$, therefore $n > 2006 \times 4 = 8024$.

The exponent of 5 in 8025! is equal to $\displaystyle\sum_{i=1}^{\infty}\left[\frac{8025}{5^i}\right] = 1605 + 321 + 64 + 12 + 2 = 2004$.

Therefore, the smallest n for which the exponent of 5 in $n!$ is equal to 2006, is 8035.

Hence the integers are 8035, 8036, 8037, 8038 and 8039.

At least partially solved by:

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