PROBLEM OF THE WEEK

Solution of Problem No. 3 (Fall 2006 Series)

Problem: Let A_1, A_2, A_3, A_4 be the areas of the faces of a tetrahedron. Let γ_{ij} be the interior angle between the faces with areas A_i and A_j. Prove that

$$A_4{}^2 = A_1{}^2 + A_2{}^2 + A_3{}^2 - 2A_1A_2\cos\gamma_{12} - 2A_2A_3\cos\gamma_{23} - 2A_3A_1\cos\gamma_{31}.$$

Solution (by Steven Landy, edited by the Panel)

Let $\vec{A}_i$ be vectors perpendicular to the sides A_i, $i = 1, 2, 3, 4$, pointing to the exterior, with length equal to the area of the corresponding face A_i. Then it is easy to see that $\vec{A}_1 + \vec{A}_2 + \vec{A}_3 + \vec{A}_4 = 0$ by representing each $\vec{A}_i$ as one half of the vector product of two edges.

Square both sides of

$$-\vec{A}_4 = \vec{A}_1 + \vec{A}_2 + \vec{A}_3$$

to get

$$\begin{aligned} A_4{}^2 = A_1{}^2 + A_2{}^2 + A_3{}^2 &+ 2A_1A_2\cos(\pi - \gamma_{12}) \\ &+ 2A_1A_3\cos(\pi - \gamma_{13}) + 2A_2A_3\cos(\pi - \gamma_{23}), \end{aligned}$$

which proves the equality.

At least partially solved by:

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