PROBLEM OF THE WEEK

Solution of Problem No. 8 (Fall 2006 Series)

Problem:

Let A be a real 3×3 skew–symmetric matrix and let S be real 3×3 symmetric. Show that the polynomial

$$p(x) = \det(A + xS)$$

has a multiple zero if and only if $p(x) = ax^3$ with same real a.

Solution (by the Panel)

Observe first that

$$\begin{aligned} f(x) &= \det(A + xS) = \det(A + xS)^T \\ &= \det(A - xS) = f(-x). \end{aligned}$$

Therefore, $f(x)$ is an odd polynomial of degree 3 or less. Thus,

$$f(x) = ax^3 + bx.$$

If $a = 0$, there is no multiple root. If $a \neq 0$, then the roots are 0, $\sqrt{-b/a}$, $-\sqrt{-b/a}$ (here, $\sqrt{y} \geq 0$ if $y \geq 0$, and $\mathrm{Im}\sqrt{y} > 0$ if $y < 0$). The only way two of them can be equal is if $b = 0$.

This proves the "only if" part. The "if" part is trivial.

At least partially solved by:

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