PROBLEM OF THE WEEK

Solution of Problem No. 10 (Fall 2007 Series)

Problem: Suppose a, b, r, s are positive numbers and $r \geq s$. Show that

$$a^r - b^r \geq \frac{r}{s} b^{r-s}(a^s - b^s),$$

and equality holds if and only if $a = b$ or $r = s$.

Solution (by Hetong Li, Freshman, Physics)

If $a \neq b$, then $\frac{a}{b} \neq 1$.

Let $f(x) = \dfrac{(\frac{a}{b})^x - 1}{x}$ $(x \neq 0)$. Then $f'(x) = \dfrac{x(\frac{a}{b})^x \ln(\frac{a}{b}) - (\frac{a}{b})^x + 1}{x^2}$.

Let $h(x) = x\left(\frac{a}{b}\right)^x \ln\left(\frac{a}{b}\right) - \left(\frac{a}{b}\right)^x + 1$.

$$\text{Then} \quad h'(x) = \left(\frac{a}{b}\right)^x \ln\left(\frac{a}{b}\right) + x\left[\ln\left(\frac{a}{b}\right)\right]^2 \left(\frac{a}{b}\right)^x - \left(\frac{a}{b}\right)^x \ln\left(\frac{a}{b}\right)$$

$$= x\left(\ln\left(\frac{a}{b}\right)\right)^2 \left(\frac{a}{b}\right)^x > 0, \text{ if } x > 0.$$

$f'(x) > 0 \;\Rightarrow\; f(x)$ strictly increases on $(0, \infty)$.

$r > s \;\Rightarrow\; f(r) > f(s) \;\Rightarrow\; \dfrac{(\frac{a}{b})^r - 1}{r} > \dfrac{(\frac{a}{b})^s - 1}{s}$

$\Rightarrow a^r - b^r > \dfrac{r}{s} \cdot b^{r-s}(a^s - b^s)$.

Update on POW 9: Solved also by Pete Kornya (Faculty, Ivy Tech).

Also solved by:

Undergraduates: Fan Zhang (So. CS)

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Others: Manuel Barbero (New York), Brian Bradie (Christopher Newport U. VA), Kunihiko Chikaya (Kunitachi, Japan), Hoan Duong (San Antonio College), Elie Ghosn (Montreal, Quebec), Pete Kornya (Faculty, Ivy Tech), Steven Landy (IUPUI Physics), Angel Plaza (ULPGC, Spain), Sorin Rubinstein (TAU faculty, Israel)