PROBLEM OF THE WEEK

Solution of Problem No. 11 (Fall 2007 Series)

Problem: Find the sum of the series

$$S = \cos^3 x - \frac{1}{3}\cos^3 3x + \frac{1}{3^2}\cos^3 3^2 x - \frac{1}{3^3}\cos^3 3^3 x + \cdots .$$

Solution (by Richard Eden, Math Grad Student)

It is easy to show, using basic trigonometric identities, that $\cos(3t) = 4\cos^3 t - 3\cos t$, so $4\cos^3 t = \cos 3t + 3\cos t$. Letting $t = 3^m x$ $(m = 0, 1, 2, \dots)$, we have

$$4\cos^3(3^m x) = \cos(3^{m+1}x) + 3\cos(3^m x).$$

Let $S_N(x) = \sum_{m=0}^{N} \frac{(-1)^m}{3^m}\cos^3(3^m x)$. Therefore,

$$\begin{aligned}
\frac{4}{3}S_N(x) &= \sum_{m=0}^{N} (-1)^m \frac{4\cos^3(3^m x)}{3^{m+1}} \\
&= \sum_{m=0}^{N} (-1)^m \left\{ \frac{\cos(3^{m+1}x)}{3^{m+1}} + \frac{\cos(3^m x)}{3^m} \right\} \\
&= \left\{ \frac{\cos(3x)}{3} + \frac{\cos(x)}{1} \right\} - \left\{ \frac{\cos(3^2 x)}{3^2} + \frac{\cos(3x)}{3} \right\} + \cdots + (-1)^N \left\{ \frac{\cos(3^{N+1}x)}{3^{N+1}} + \frac{\cos(3^N x)}{3^N} \right\} \\
&= (-1)^N \frac{\cos(3^{N+1}x)}{3^{N+1}} + \cos x.
\end{aligned}$$

The first term approaches 0 as $N \to \infty$. Thus,

$$\begin{aligned}
S &= \cos^3 x - \frac{1}{3}\cos^3(3x) + \frac{1}{3^2}\cos^3(3^2 x) - \frac{1}{3^3}\cos^3(3^3 x) + \dots \\
&= \sum_{m=0}^{\infty} \frac{(-1)^m}{3^m}\cos^3(3^m x) \\
&= \lim_{N\to\infty} S_N(x) \\
&= \frac{3}{4}\cos x.
\end{aligned}$$

Also solved by:

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