PROBLEM OF THE WEEK

Solution of Problem No. 12 (Fall 2007 Series)

Problem: At each generation a microbe either splits into two perfect copies of itself or dies. If the probability of splitting is p, what is the probability that a single microbe will produce an everlasting colony?

Solution (by Dr. Sorin Rubinstein, Tel Aviv, Israel)

Let q be the probability that a single microbe will produce an everlasting colony. This probability equals the probability of splitting multiplied by the probability that at least one of the descendants will produce an everlasting colony. It means that q satisfies the equation:

$$q = p(1-(1-q)^2).$$

This equation may be rewritten: $pq^2 + (1-2p)q = 0$ and has the solution $q = 0$ if $p = 0$, and the solutions $q^{(1)} = 0$ and $q^{(2)} = \frac{2p-1}{p}$ if $0 < p \leq 1$. If $0 < p \leq \frac{1}{2}$ then $q^{(2)} = \frac{2p-1}{p} \leq 0$ and $q = 0$ is the only admissible solution. Assume that $\frac{1}{2} < p \leq 1$. Then $q^{(2)} = \frac{2p-1}{p} > 0$.

Let q_n be the probability that a single microbe will produce a colony which will last at least n generations. Then $q_0 = 1$ and q_n is a decreasing sequence that converges to q. Moreover q_{n+1} equals the probability that the given microbe splits multiplied by the probability that at least one of its descendants will produce a colony which will last at least n generations: $q_{n+1} = p(1-q_n)^2)$.

Evidently $q_0 = 1 \geq \frac{2p-1}{p}$. Assume that $q_n \geq \frac{2p-1}{p}$ for some n. Then one obtains: $0 \leq 1-q_n \leq 1-\frac{2p-1}{p} = \frac{1-p}{p}$ and therefore that $(1-q_n)^2 \leq \frac{(1-p)^2}{p^2}$. From this follows that:

$$q_{n+1} = p(1-(1-q_n)^2) \geq p\left(1-\frac{(1-p)^2}{p^2}\right) = \frac{2p-1}{p}.$$

Hence $q_n \geq \frac{2p-1}{p}$ for every n and therefore $q \geq \frac{2p-1}{p} > 0$. It follows that $q = q^{(2)}$ for $\frac{1}{2} < p \leq 1$. Hence the probability that a single microbe will produce an everlasting colony is 0 if $0 \leq p \leq \frac{1}{2}$ and $\frac{2p-1}{p}$ if $\frac{1}{2} < p \leq 1$.

Also solved by:

Undergraduates: Lokesh Batra (Fr. Engr), Noah Blach (Fr. Math), Siddharth Tekriwal (So. Engr.)

Others: Aviv Adler (College Prep HS, CA), Brian Bradie (Christopher Newport U. VA), Elie Ghosn (Montreal, Quebec), Pete Kornya (Faculty, Ivy Tech)

There were three other people who found the correct answer without giving a sufficiently complete proof.