PROBLEM OF THE WEEK

Solution of Problem No. 13 (Fall 2007 Series)

Problem: Show that if

$$\cos(2^n x) > \cos(2^n y) \quad \text{for all non–negative integers } n,$$

where x and y are real numbers, then $x = 2\pi k$ for some integer k.

Solution (by Pete Kornya, Ivy Tech faculty, Bloomington, IN)

Using the identity $\cos 2\theta = 2\cos^2\theta - 1$, we have
$\cos(2^{n+1}x) > \cos(2^{n+1}y) \Rightarrow \cos^2(2^n x) > \cos^2(2^n y)$. Since also $\cos(2^n x) > \cos(2^n y)$, it follows that $\cos(2^n x) > 0$ for all $n \geq 0$.

Consider the binary expansion $\dfrac{x}{2\pi} - \left[\dfrac{x}{2\pi}\right] = \sum\limits_{i=1}^{\infty} \delta_i 2^{-i}$ with each $i = 0$ or 1. We may assume that there are not repeating 1's from some place m on, since $\sum\limits_{i=m}^{\infty} 2^{-i} = 2^{-m+1}$. Therefore if some $\delta_i = 1$ then there is also a $\delta_n = 1$ followed by a $\delta_{n+1} = 0$. Then

$$\begin{aligned}
\cos(2^{n-1}x) &= \cos\left(2^{n-1}\cdot 2\pi\left(\frac{x}{2\pi} - \left[\frac{x}{2\pi}\right]\right) + 2^{n-1}\cdot\left[\frac{x}{2\pi}\right]2\pi\right) \\
&= \cos\left(2^{n-1}\cdot 2\pi\cdot\sum_{i=1}^{\infty}\delta_i 2^{-i}\right) \\
&= \cos\left(\pi + \sum_{i=n+2}^{\infty}\delta_i 2^{-i+n}\pi\right).
\end{aligned}$$

But then, since $\pi \leq \pi + \sum\limits_{i=n+2}^{\infty}\delta_i 2^{-i+n}\pi < \dfrac{3\pi}{2}$, we would have $\cos 2^{n-1}x \leq 0$, a contradiction. Therefore $\delta_i = 0$ for all i. Then $\dfrac{x}{2\pi} - \left[\dfrac{x}{2\pi}\right] = 0$ and $x = 2\pi\left[\dfrac{x}{2\pi}\right]$ as required.

Also solved by:

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