PROBLEM OF THE WEEK

Solution of Problem No. 14 (Fall 2007 Series)

Problem: The three vertices of a triangle have integer coordinates and lie on a circle of radius R. If the side lengths are a, b, c, show that $abc \geq 2R$. Can equality hold?

Solution #1 (by Siddharth Tekriwal, Sophomore, Mechanical Engr.)

Let θ be the angle opposite side a. Then the area of the triangle is:

$$A = \frac{1}{2} bc \sin\theta.$$

Also, the side a subtends an angle 2θ at the centre of the circum–circle. So

$$a = 2R\sin\theta.$$

Hence $A = \dfrac{abc}{4R}$. From Pick's theorem, the area of any (non–self–intersecting) polygon whose vertices are lattice points is $\left(\dfrac{v}{2} + i - 1\right)$, where v is the number of lattice points on the perimeter and i is the number of lattice points inside the polygon. Since for a triangle, $v \geq 3$, and $i \geq 0$, we have area at least $\dfrac{3}{2} - 1 = \dfrac{1}{2}$. So, $A \geq \dfrac{1}{2}$ or, $\dfrac{abc}{4R} \geq \dfrac{1}{2}$ or, $abc \geq 2R$.

A different argument for the second part of the proof and the equality question was given by several people. Excerpt from the solution by Brian Huang (Jr. Saint Joseph's HS, South Bend, IN):

$$A = \frac{1}{2}\left\|\begin{matrix} x_A & x_B & x_C \\ y_A & y_B & y_C \\ 1 & 1 & 1 \end{matrix}\right\|$$

Since $A > 0$

$$\left\|\begin{matrix} x_A & x_B & x_C \\ y_A & y_B & y_C \\ 1 & 1 & 1 \end{matrix}\right\| > 0$$

but since the coordinates are integers, the absolute value of the determinant must be an integer value. Thus

$$A = \frac{1}{2}\left\|\begin{matrix} x_A & x_B & x_C \\ y_A & y_B & y_C \\ 1 & 1 & 1 \end{matrix}\right\| \geq \frac{1}{2}.$$

Let $\triangle ABC : (0,0), (1,0), (0,1)$

$$a = 1, \quad b = \sqrt{2}, \quad c = 1, \quad R = \frac{\sqrt{2}}{2}.$$

Thus, equality can hold.

Also solved by:

Graduates: Richard Eden (Math)

Others: Aviv Adler (Jr. College Prep. HS, CA), Manuel Barbero (New York), Brian Bradie (Christopher Newport U. VA), Kunihiko Chikaya (Kunitachi, Japan), Subham Ghosh (Washington Univ. St. Louis), Elie Ghosn (Montreal, Quebec), Pete Kornya (Faculty, Ivy Tech), Steven Landy (IUPUI Physics), Graeme McRae (Palmdale CA), Sorin Rubinstein (TAU faculty, Israel)

A correct solution was submitted by an unsigned person.