PROBLEM OF THE WEEK
Solution of Problem No. 3 (Fall 2007 Series)

Problem: For any real positive number r, let $\{r\}$ define the integer closest to r (for $r = k + \frac{1}{2}, k \in \mathbb{N}$, choose $\{r\} = k$). Evaluate the sum

$$\sum_{k=1}^{\infty} \left\{\sqrt{k}\right\}^{-3}.$$

You may use: $\sum_{k=1}^{\infty} k^{-2} = \pi^2/6$.

Solution (by Stephen Casey, University College Cork, Ireland)

Let $S_n = \{k \in \mathbb{Z}^+ : \{\sqrt{k}\} = n\}$ for each positive integer n.
Then,

$$\begin{aligned} k \in S_n \quad &\Leftrightarrow \quad n - \frac{1}{2} < \sqrt{k} \le n + \frac{1}{2} \\ &\Leftrightarrow \quad \left(n - \frac{1}{2}\right)^2 < \left(\sqrt{k}\right)^2 \le \left(n + \frac{1}{2}\right)^2 \qquad \text{[since everything is positive]} \\ &\Leftrightarrow \quad n^2 - n + \frac{1}{4} < k \le n^2 + n + \frac{1}{4} = n^2 - n + \frac{1}{4} + 2n \end{aligned}$$

Hence,

$$S_n = \{n^2 - n + 1, n^2 - n + 2, \ldots, n^2 - n + 2n\}$$

and

$$|S_n| = 2n$$

$$\begin{aligned} \Longrightarrow \qquad \sum_{k=1}^{\infty} \left\{\sqrt{k}\right\}^{-3} &= \sum_{n=1}^{\infty} \sum_{k \in S_n} \left\{\sqrt{k}\right\}^{-3} \\ &= \sum_{n=1}^{\infty} (2n)(n^{-3}) \\ &= 2 \sum_{n=1}^{\infty} n^{-2} \\ &= 2\left(\frac{\pi^2}{6}\right) = \frac{\pi^2}{3} \end{aligned}$$

$$\sum_{k=1}^{\infty} \left\{\sqrt{k}\right\}^{-3} = \frac{\pi^2}{3}$$

Update on POW 1 & 2: Solved also by Noah Blach (Fr. Math).

Also solved by:

Undergraduates: Lokesh Batra (Fr. Engr), Alan Bernstein (Sr. ECE), Noah Blach (Fr. Math), Hetong Li (Fr. Science)

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