PROBLEM OF THE WEEK

Solution of Problem No. 4 (Fall 2007 Series)

Problem: Let $P(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$, where $a_0, \ldots, a_n$ are integers. Show that if P takes the value 2007 for four distinct integral values of x, then P cannot take the value 1990 for any integral value of x. (Partial credit if you can prove it with "four" replaced by "five".)

Solution (by Angel Plaza, ULPGC, Spain)

Let us consider the polynomial $Q(x) = P(x) - 2007$. Since P takes the value 2007 for four distinct integral values of x, then Q has at least four different integral roots: $r_1, \ldots, r_4$. $Q(x) = (x - r_1)(x - r_2)(x - r_3)(x - r_4)R(x)$, where R is also a polynomial with integral coefficients.

Let us suppose that there is an integer x^* such that $P(x^*) = 1990$, then $Q(x^*) = -17$, that is $(x^* - r_1)(x^* - r_2)(x^* - r_3)(x^* - r_4)R(x^*) = -17$. Since by hypothesis $r_1, \ldots, r_4$ are all different $(x^* - r_1)(x^* - r_2)(x^* - r_3)(x^* - r_4)$ are four different divisors of -17. But the only divisors of -17 are $1, -1, 17, -17$. Hence $1(-1)(17)(-17)R(x^*) = -17$, which implies $R(x^*) = \dfrac{1}{17}$. This contradicts the fact that $R(x^*)$ is an integer.

Also solved by:

Undergraduates: Alan Bernstein (Sr. ECE), Noah Blach (Fr. Math)

Graduates: Chi Weng Cheong (Math), David Lomiashvili (Phys.)

Others: Manuel Barbero (New York), Brian Bradie (Christopher Newport U. VA), Stephen Casey (Ireland), Hoan Duong (San Antonio College), Patricia Johnson (OSU–Lima, OH), Pete Kornya (Faculty, Ivy Tech), Steven Landy (IUPUI Physics), Graeme McRae, Matias Victor Moya Giusti (Sr. Univ. de Córdoba), Sorin Rubinstein (TAU faculty, Israel)

A correct solution for the partial credit version was submitted by Subham Ghosh.