PROBLEM OF THE WEEK

Solution of Problem No. 5 (Fall 2007 Series)

Problem: Let a particle move on a straight line with non–decreasing acceleration for $0 \le t \le T$. Show that its velocity at $t = \frac{1}{2}T$ cannot exceed its average velocity.

The shortest solutions invoke the theory of convex functions (i.e., functions that are concave upward), applied to the velocity function. The solution presented uses the same underlying idea without invoking the theory.

Solution (by Pete Kornya, Ivy Tech faculty, Bloomington, IN)

Let $v(t), a(t)$ be the velocity, respectively the acceleration of the particle at time t. Since a is nondecreasing, if $\frac{T}{2} \le t \le T$ then

$$\int_{s=T/2}^{t} a(s)ds \ge \int_{s=T/2}^{t} a(T/2)ds = (t - T/2)a(T/2) \tag{1}$$

If $0 \le t \le T/2$ then $\displaystyle\int_{s=T/2}^{t} a(s)ds = -\int_{s=t}^{T/2} a(s)ds \ge -\int_{s=t}^{T/2} a(T/2)ds = (t - T/2)a(T/2)$ as well. Therefore the inequality (1) is true for all $0 \le t \le T$. Then

$$\begin{aligned}
\frac{1}{T}(\text{net distance travelled}) = \frac{1}{T}\int_{t=0}^{T} v(t)dt &= \frac{1}{T}\int_{t=0}^{T}\left[v(T/2) + \int_{s=T/2}^{t} a(s)ds\right]dt \\
&\ge \frac{1}{T}\int_{t=0}^{T}\left[\, v(T/2) + (t - T/2)a(T/2)\,\right]\, dt \\
&= v(T/2)
\end{aligned}$$

Also solved by:

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