PROBLEM OF THE WEEK

Solution of Problem No. 6 (Fall 2007 Series)

Problem: Show that the integer nearest to $\dfrac{n!}{e}$ $(n \geq 2)$ is divisible by $n-1$ but not by n.

Solution (by Elie Ghosn, Montreal, Quebec)

We have $e^{-1} = \sum_{k=0}^{\infty} \dfrac{(-1)^k}{k!}$. Therefore,

$$\frac{n!}{e} = n!e^{-1} = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} + n! \sum_{k=n+1}^{\infty} \frac{(-1)^k}{k!}.$$

The first term is obviously an integer and the second term can be bounded by (remainder of an alternating series)

$$\left| n! \sum_{k=n+1}^{\infty} \frac{(-1)^k}{k!} \right| \leq n! \cdot \frac{1}{(n+1)!} = \frac{1}{n+1} \leq \frac{1}{3} \quad \text{since} \quad n \geq 2.$$

Therefore $n! \sum_{k=0}^{n} \dfrac{(-1)^k}{k!}$ is the nearest integer to $\dfrac{n!}{e}$. This integer is not divisible by n because:

$$n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} = n \cdot \left[(n-1)! \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} \right] + (-1)^n$$

and it is divisible by $(n-1)$ because:

$$\begin{aligned} n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} &= n(n-1) \left[(n-2)! \sum_{k=0}^{n-2} \frac{(-1)^k}{k!} \right] + (-1)^{n-1} \cdot n + (-1)^n \\ &= (n-1) \left\{ n \left[(n-2)! \sum_{k=0}^{n-2} \frac{(-1)^k}{k!} \right] + (-1)^{n-1} \right\}, \end{aligned}$$

since terms between square bracket are obviously integers.

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