PROBLEM OF THE WEEK

Solution of Problem No. 9 (Fall 2007 Series)

Problem: Show that $\displaystyle\sum_{k=1}^{n} \frac{1}{k}\left(\binom{n}{k}+1\right) = \sum_{k=1}^{n} \frac{2^k}{k}.$

This problem is identical to Problem #10 from Fall 2006. The panel apologizes for the duplication and thanks Steve Spindler and Nate Orlow for pointing it out. This problem will not be counted for the semester's competition. The solution provided below is different from the previously published one.

Solution (by Noah Blach, Freshman, Math)

Let $\displaystyle U_n = \sum_{k=1}^{n}\left(\binom{n}{k}+1\right)\frac{1}{k}$

$$\begin{aligned}
U_{n+1} - U_n &= \sum_{k=1}^{n+1}\left(\binom{n+1}{k}+1\right)\frac{1}{k} - \sum_{k=1}^{n}\left(\binom{n}{k}+1\right)\frac{1}{k} \\
&= \frac{2}{n+1} + \sum_{k=1}^{n}\left(\binom{n+1}{k}-\binom{n}{k}\right)\frac{1}{k} = \frac{2}{n+1} + \sum_{k=1}^{n}\binom{n}{k-1}\frac{1}{k} \\
&= \frac{2}{n+1} + \sum_{k=1}^{n}\frac{n!}{(k-1)!\cdot k\cdot(n-k+1)!} = \frac{2}{n+1} + \sum_{k=1}^{n}\frac{(n+1)!}{k!(n-k+1)!}\cdot\frac{1}{n+1} \\
&= \frac{2}{n+1} + \frac{1}{n+1}\sum_{k=1}^{n}\binom{n+1}{k} = \frac{2}{n+1} + \frac{1}{n+1}\left(2^{n+1}-2\right) = \frac{2^{n+1}}{n+1}
\end{aligned}$$

$$\begin{aligned}
U_1 = \frac{2}{1} = 2^1, \quad \text{and} \quad U_n &= U_1 + \sum_{k=2}^{n}\left(U_k - U_{k-1}\right) = U_1 + \sum_{k=2}^{n}\frac{2^k}{k} \\
&= \sum_{k=1}^{n}\frac{2^k}{k}.
\end{aligned}$$

Also solved by:

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