PROBLEM OF THE WEEK

Solution of Problem No. 10 (Fall 2008 Series)

Problem: Find all differentiable functions $f : [a, b] \to \mathbb{R}$ which have the property that

$$\int_{\alpha}^{\beta} f(x)dx = \frac{f(\alpha) + f(\beta)}{2}(\beta - \alpha), \tag{1}$$

whenever $a \le \alpha < \beta \le b$.

Solution (by Brian Bradie, Christopher Newport University, VA)

Let $f : [a, b] \to \mathbb{R}$ be differentiable and suppose f satisfies (1) whenever $a \le \alpha < \beta \le b$. Differentiating (1) with respect to β yields

$$f(\beta) = \frac{f(\alpha) + f(\beta)}{2} + \frac{1}{2}f'(\beta)(\beta - \alpha), \tag{2}$$

while differentiating (1) with respect to α yields

$$-f(\alpha) = -\frac{f(\alpha) + f(\beta)}{2} + \frac{1}{2}f'(\alpha)(\beta - \alpha). \tag{3}$$

If we subtract (3) from (2) we find

$$f(\alpha) + f(\beta) = f(\alpha) + f(\beta) + \frac{1}{2}(\beta - \alpha)(f'(\beta) - f'(\alpha)),$$

which simplifies to

$$f'(\beta) = f'(\alpha) \tag{4}$$

given that $\alpha < \beta$. As (4) holds whenever $a \le \alpha < \beta \le b$, it follows that f' is constant along $[a, b]$. Thus, if $f : [a, b] \to \mathbb{R}$ is a differentiable function which satisfies (1) whenever $a \le \alpha < \beta \le b$, then f is a linear function; that is, $f(x) = mx + c$ for some constants m and c.

Note that if we know f is at least twice continuously differentiable, then we may use the fact that the formula on the right–hand side of (1) is the trapezoidal rule, so

$$\int_{\alpha}^{\beta} f(x)dx - \frac{f(\alpha) + f(\beta)}{2}(\beta - \alpha) = \frac{(\beta - \alpha)^3}{12}f''(\xi),$$

where $\alpha < \xi < \beta$. Thus, (1) holds whenever $a \le \alpha < \beta \le b$ if and only if $f''(x) \equiv 0$; that is, $f(x) = mx + c$ for some constants m and c.

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