## PROBLEM OF THE WEEK

Solution of Problem No. 13 (Fall 2008 Series)

Problem: At time 0 each of the positions $1,2, \ldots, n$ on the real line is occupied by a robot, and position 0 is occupied by the prey. At time $k(k=1, \ldots, n)$ one of the robots, selected at random, jumps one unit to the left, unless that robot has been previously disabled, in which case nothing happens. If it lands on position 0 , the prey is destroyed; but if it lands on another robot, both robots are disabled. Assuming that each robot is selected to jump exactly once and that all $n$ ! jumping orders are equally likely, find the probability $p_{n}$, that the prey is eventually destroyed and also find $\lim _{n \rightarrow \infty} p_{n}$. (Your answer for $p_{n}$ need not be in closed form.)

This problem was proposed by Dan Brown of Electronic Arts. His solution is the same as the first one given below. The second solution is more typical of the correct solutions received.

Solution 1 (by Sorin Rubinstein, TAU faculty, Israel)

For any $k$ with $1 \leq k \leq n$ we denote by $D_{k}$ the number of permutations of $\{1,2,3, \ldots, n\}$ which contain the decreasing subsequence (of which the elements are not necessarily on consecutive positions in the original permutation): $k, k-1, \ldots, 1$. In other words, the robots occupying positions $1,2, \ldots, k$ jump in reverse order. There are $\binom{n}{k}$ possibilities to chose the positions of $k, k-1, \ldots, 1$ and for any such choice there are $(n-k)$ ! possibilities to fill in the remaining places. Thus:

$$
D_{k}=\binom{n}{k}(n-k)!=\frac{n!}{k!} .
$$

Moreover we define $D_{k}=0$ for $k>n$.
The prey will be destroyed if and only if for some $k$ the robots are chosen according to a permutation of the set $\{1,2,3, \ldots, n\}$ for which $2 k-1,2 k-2, \ldots, 1$ is a subsequence but $2 k, 2 k-1, \ldots, 1$ is not.
For each $k$ with $1 \leq k \leq\left\lceil\frac{n}{2}\right\rceil$ there are $D_{2 k-1}-D_{2 k}$ such permutations. (Here and elsewhere $[x]$ represents the least integer which is greater than or equal to $x$ ) Thus:

$$
p_{n}=\frac{1}{n!} \sum_{k=1}^{\left\lceil\frac{n}{2}\right\rceil}\left(D_{2 k-1}-D_{2 k}\right)=\frac{1}{1!}-\frac{1}{2!}+\frac{1}{3!}-\frac{1}{4!}+\cdots+\frac{(-1)^{n+1}}{n!}=\sum_{k=1}^{n} \frac{(-1)^{k+1}}{k!}
$$

Therefore:

$$
\lim _{n \rightarrow \infty} p_{n}=\frac{1}{1!}-\frac{1}{2!}+\frac{1}{3!}-\frac{1}{4!}+\cdots=1-\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\cdots\right)=1-\frac{1}{e}
$$

Solution 2 (by Steve Spindler, Chicago)

We establish the following iterative formula for $p_{n}$ :

$$
p_{n}=\left(\frac{1}{n}\right)\left(p_{n-2}+p_{n-3}+\cdots+p_{2}+p_{1}+1\right)
$$

For $3 \leq k \leq n$, if the $k$-th robot jumps first, it disables the $(k-1)$-st robot and itself. The actions of robots $k+1$ through $n$ are irrelevant, so the probability of destroying the prey is then $p_{k-2}$. If the second robot jumps first, it disables the first robot and the prey cannot be destroyed, so the probability is zero. And if the first robot jumps first, the prey is destroyed and the probability is clearly one. The probability or any robot being selected is $\frac{1}{n}$, so multiplying and summing for robots $1,2, \ldots, n$ gives the result above for the total probability.
We can rewrite this expression as: $n p_{n}=p_{n-2}+p_{n-3}+\cdots+p_{2}+p_{1}+1$. Subtracting the analogous expression for $(n-1) p_{n-1}$ gives: $n p_{n}-(n-1) p_{n-1}=p_{n-2}$.
Now we can prove inductively that $p_{n}=\sum_{k=1}^{n} \frac{(-1)^{k+1}}{k!}$. It is clearly true for $n=1$ and $n=2$. Assume it is true for $k<n$; then

$$
\begin{aligned}
n p_{n} & =(n-1) p_{n-1}+p_{n-2}=(n-1) \sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{k!}+\sum_{k=1}^{n-2} \frac{(-1)^{k+1}}{k!} \\
& =n \sum_{k=1}^{n-2} \frac{(-1)^{k+1}}{k!}+\frac{(n-1)(-1)^{n}}{(n-1)!} \\
& =n \sum_{k=1}^{n-2} \frac{(-1)^{k+1}}{k!}+\frac{n(-1)^{n}}{(n-1)!}-\frac{(-1)^{n}}{(n-1)!} \\
& =n \sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{k!}+\frac{n(-1)^{n+1}}{n!}=n \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k!} .
\end{aligned}
$$

As desired. Finally, $\lim _{n \rightarrow \infty} p_{n}=\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!}=1-\sum_{k=0}^{\infty} \frac{(-1)^{k}}{k!}=1-e^{-1}$.

The problem was also solved by:

Undergraduates: David Elden (So. Mech.E)
Graduates: Huanyu Shao (CS)
Others: Brian Bradie (Christopher Newport U. VA), Randin Divelbiss (Undergraduate, University of Wisconsin-Stevens Point), Elie Ghosn (Montreal, Quebec), Steven Landy (IUPUI Physics staff), Wei-hsiang Lien (Grad student, National Chiao-Tung Univ, Taiwan), Peyman Tavallali (Grad. student, NTU, Singapore)

