PROBLEM OF THE WEEK

Solution of Problem No. 3 (Fall 2008 Series)

Problem: Show that

$$n^k = \sum_{l=1}^{k} (-1)^{k-l} \binom{n}{l} \binom{n-1-l}{k-l} l^k, \quad \text{where}$$

n and k are positive integers and $n \geq k+1$.

Solution (by Elie Ghosn, Montreal, Quebec)

The Lagrange interpolating polynomial of degree $\leq k$ (k positive integer) that passes through the $(k+1)$ points $(l, l^k)_{l=0,1,\ldots,k}$ is given by:

$$p(x) = \sum_{l=1}^{k} \pi_l(x) l^k \quad \text{where} \quad \pi_l(x) = \prod_{\substack{j=0 \\ j \neq l}}^{k} \frac{(x-j)}{(l-j)}$$

$p(x)$ is equal to $Q(x) = x^k$ since both are of degree $\leq k$ and $p(l) = Q(l)$ for $l = 0, \ldots, k$. Therefore for $x = n \geq k+1$, n integer, we have:

$$n^k = \sum_{l=1}^{k} \left(\prod_{\substack{j=0 \\ j \neq l}}^{k} \frac{(n-j)}{(l-j)} \right) l^k$$

but

$$\prod_{\substack{j=0 \\ j \neq l}}^{k} (n-j) = \frac{\prod_{j=0}^{k} (n-j)}{n-l} = \frac{n!}{(n-k-1)!(n-l)} = \frac{n!}{(n-l)!} \cdot \frac{(n-l-1)!}{(n-k-1)!}$$

and

$$\prod_{\substack{j=0 \\ j \neq l \\ 1 \leq l \leq k}}^{k} (l-j) = \prod_{j=0}^{l-1} (l-j) \prod_{j=l+1}^{k} (l-j) = l!\, (-1)^{k-l} (k-l)!.$$

Therefore,

$$\begin{aligned} n^k &= \sum_{l=1}^{k} \frac{n!}{(n-l)!} \, \frac{(n-l-1)!}{(n-k-1)!} \, \frac{(-1)^{k-l}}{l!(k-l)!} l^k \\ &= \sum_{l=1}^{k} (-1)^{k-l} \binom{n}{l} \binom{n-1-l}{k-l} l^k. \end{aligned}$$

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