PROBLEM OF THE WEEK
Solution of Problem No. 5 (Fall 2008 Series)

Problem: Evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{1^2+2^2+\cdots+n^2}$.

This problem was proposed by Brian Bradie of Christopher Newport University.

Solution (by Richard B. Eden, Math graduate student, Purdue Univ.)

Let $S = \displaystyle\sum_{n=1}^{\infty} a_n$ denote the given sum. Since $1^2+2^2+\cdots+n^2 = \dfrac{n(n+1)(2n+1)}{6}$, then $a_n = \dfrac{6(-1)^{n-1}}{n(n+1)(2n+1)}$. Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges, then S converges. By partial fractions, $S = 6\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\left[\frac{1}{n}+\frac{1}{n+1}-\frac{4}{2n+1}\right]$.

Let $T_k = \displaystyle\sum_{n=1}^{k}(-1)^{n-1}\left[\frac{1}{n}+\frac{1}{n+1}\right]$. Then

$$T_k = \left[\frac{1}{1}+\frac{1}{2}\right] - \left[\frac{1}{2}+\frac{1}{3}\right] + \cdots + (-1)^{k-1}\left[\frac{1}{k}+\frac{1}{k+1}\right] = 1 + (-1)^{k-1}\frac{1}{k+1}.$$

So $\displaystyle\lim_{k\to\infty} T_k = 1$. From calculus, we have $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$. *

Therefore,

$$S = 6\left[\lim_{k\to\infty} T_k + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{2n+1}\right] = 6\left[1+4\left(\frac{\pi}{4}-1\right)\right] = 6\pi - 18.$$

* This is because $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\, x^{2n+1} = \arctan x$ for $|x| \le 1$.

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