PROBLEM OF THE WEEK

Solution of Problem No. 1 (Fall 2009 Series)

Problem: For each odd positive integer n, show that

$$1 \cdot 3 \cdot 5 \cdots (2n-1) + 2 \cdot 4 \cdot 6 \cdots 2n$$

is divisible by $2n+1$.

Solution (by Brent Woodhouse, Freshman, Purdue University)

We evaluate the given expression modulo $2n+1$. For $k = 1, 2, \ldots, n$, note that

$$2k \equiv -(2n+1-2k) = -(2n-(2k-1)) \pmod{2n+1}.$$

Replacing each of the n even integers of the form $2k$ in the product $2 \cdot 4 \cdot 6 \cdots 2n$ with the corresponding negative odd integer $-(2n-(2k-1))$ to which it is congruent yields the following:

$$\begin{aligned}
&1 \cdot 3 \cdot 5 \cdots (2n-1) + 2 \cdot 4 \cdot 6 \cdots 2n \\
&\equiv 1 \cdot 3 \cdot 5 \cdots (2n-1) + (-(2n-1))(-(2n-3)(-(2n-5)) \cdots (-5)(-3)(-1) \pmod{2n+1} \\
&\equiv 1 \cdot 3 \cdot 5 \cdots (2n-1) + (-1)^n(1 \cdot 3 \cdot 5 \cdots (2n-1)) \pmod{2n+1}
\end{aligned}$$

Because n is odd, $(-1)^n = -1$ and

$$\begin{aligned}
&1 \cdot 3 \cdot 5 \cdots (2n-1) + 2 \cdot 4 \cdot 6 \cdots 2n \\
&\equiv 1 \cdot 3 \cdot 5 \cdots (2n-1) - 1 \cdot 3 \cdot 5 \cdots (2n-1) \equiv 0 \pmod{2n+1}.
\end{aligned}$$

Thus the given expression is divisible by $2n+1$.

The problem was also solved by:

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