PROBLEM OF THE WEEK

Solution of Problem No. 11 (Fall 2009 Series)

Problem: Given positive integers a and b, with $a > b$, a positive integer n is called attainable if n can be written in the form $xa + yb$ with x and y nonnegative integers. If there are exactly 35 non–attainable positive integers, one of which is 58, find a and b.

Solution (by Tom Engelsman, Chicago, IL)

It is desired to express the # of unattainable numbers as a function of a and b.

(i) $\gcd\{a, b\} = 1$, or else an infinite number of unattainable numbers exists.

(ii) If $x > a \cdot b$, then x is attainable since a and b are relatively prime, which yields ay such that $0 \leq y < b$ and $ay \equiv x (\bmod b)$. Then $x = ay + bn$ for $y \cdot n \geq 0$.

(iii) One must count the # of unattainable numbers between 1 and $a \cdot b$.

For $0 \leq i < b$, there are $\left\lceil \frac{ab - ai}{b} \right\rceil$ attainable numbers between 1 and $a \cdot b$ that can be expressed as $ai + bj$ between 0 and $ab - 1$, or just:

$$\sum_{i=0}^{b-1} \left\lceil \frac{ab - ai}{b} \right\rceil = \sum_{i=0}^{b-1} \left\lceil a - \frac{ai}{b} \right\rceil = \sum_{i=0}^{b-1} a - \left\lfloor \frac{ai}{b} \right\rfloor .$$

So # of attainable numbers $= ab - \sum_{i=0}^{b-1} \left\lfloor \frac{ai}{b} \right\rfloor$.

This in turn makes the # of unattainable numbers, call it S_u, equal to:

$$S_u = \sum_{i=0}^{b-1} \left\lfloor \frac{ai}{b} \right\rfloor .$$

If $\{x\} = x - \lfloor x \rfloor$ for $x \in R$, then:

$$\frac{a(b-1)}{2} = \sum_{i=0}^{b-1} \frac{ai}{b} = \sum_{i=0}^{b-1} \left\lfloor \frac{ai}{b} \right\rfloor + \left\{ \frac{ai}{b} \right\}.$$

Thus

$$\frac{a(b-1)}{2} = S_u + \sum_{i=0}^{b-1} \left\{ \frac{ai}{b} \right\}.$$

Since a and b are relatively prime, the quantity $\left\{\dfrac{ai}{b}\right\}$ takes on:

$$0, \frac{1}{b}, \frac{2}{b}, \ldots, \frac{b-1}{b} \quad \text{for } i = 1, 2, \ldots, b-1$$

which leads to:

$$\frac{a(b-1)}{2} = S_u + \sum_{i=0}^{b-1} \frac{i}{b} = S_u + \frac{b-1}{2}$$
$$S_u = \frac{(a-1)(b-1)}{2}.$$

If $S_u = 35$, then $(a-1)(b-1) = 70$ which yields:

$$(a, b) = (71, 2), (36, 3), (15, 6), (11, 8).$$

The pairs $(36, 3)$ and $(15, 6)$ are eliminated since they aren't relatively prime, and the pair $(71, 2)$ yields $58 = (71)(0) + (2)(29)$ and is also eliminated. Hence: $a = 11, b = 8$.

The problem was also solved by:

Undergraduates: Clara Bennett (Phys), Kun–Chieh Wang (Sr. Math)

Graduates: Richard Eden (Math), Tairan Yuwen (Chemistry)

Others: Manuel Barbero (New York), Gruian Cornel (IT, Romania), Elie Ghosn (Montreal, Quebec), Steven Landy (IUPUI Physics staff), Kevin Laster (Indianapolis, IN), Sorin Rubinstein (TAU faculty, Israel), Craig Schroeder (Grad student, Stanford Univ.) Steve Spindler (Chicago), Amitabha Tripathi (SUNY, NY)