PROBLEM OF THE WEEK

Solution of Problem No. 14 (Fall 2009 Series)

Problem: A set F is called countable if either F is finite or there is a one–to–one correspondence between the elements of F and the natural numbers. Two sets A and B are called almost–disjoint if $A \cap B$ is finite.

Prove or disprove: There are uncountably many pairwise almost–disjoint sets of natural numbers (positive integers). In more formal language: Does there exist an uncountable set F such that each element of F is a set of natural numbers and each two elements of F are almost–disjoint?

Solution (by Thierry Zell, Hickory, NC)

Let $I = (0.1, 1)$, and associate to each $x \in I$ the subset:

$$A_x = \{\lfloor 10^n x \rfloor \mid n \in \mathbb{Z}, n \geq 1\}$$

Each subset A_x is an infinite subset of the natural numbers. Through our choice of I, each set A_x contains exactly one n–digit element for all $n \geq 1$, which represents the first n decimals of x in base 10. If x and y are two distinct elements of I, their decimal expansion must first differ at some rank N; we then have $|A_x \cap A_y| = N - 1$.

Thus, the collection of subsets $\{A_x \mid x \in I\}$ is an uncountable collection of pairwise almost–disjoint sets.

The problem was also solved by:

Graduates: Rodrigo Ferraz de Andrade (Math), Tairan Yuwen (Chemistry)

Others: Elie Ghosn (Montreal, Quebec), Steven Landy (IUPUI Physics staff), Kevin Laster (Indianapolis, IN), Sorin Rubinstein (TAU faculty, Israel), Craig Schroeder (Grad student, Stanford Univ.)