PROBLEM OF THE WEEK

Solution of Problem No. 3 (Fall 2009 Series)

Problem: Suppose that $a_1, a_2, \ldots, a_n$ are real numbers. obviously if they are all positive, then the n sums

$$\sum_i a_i, \sum_{i<j} a_i a_j, \sum_{i<j<k} a_i a_j a_k, \ldots, a_1 a_2 \ldots a_n$$

are all positive. Prove that the converse is also true.

Solution (by Mark Sellke, Klondike Middle School, Indiana)

Let $\sum a_i = s_1, \sum_{i,j} a_i a_j = s_2$, etc. Note that the s_i's correspond to the coefficients of a polynomial of degree n with roots $a_1, a_2, \ldots, a_n : p(x) = x^n - x^{n-1}s_1 + x^{n-2}s_2 - \cdots$. As each $s_i > 0$, the coefficients have alternating signs. Thus, no negative r can be a root, as $p(r)$ has terms all of the same sign for $r < 0$. Also, $r \neq 0$, as the constant term of the polynomial, $\pm s_n$, is non–zero. Thus, all real roots are positive , so all a_i's are positive, as desired.

The problem was also solved by:

Undergraduates: Kilian Cooley (Fr.), Artyom Melanich (Fr. Engr.), Brent Woodhouse (Fr. Science)

Graduates: Richard Eden (Math), Vitezslav Kala (Math), Xin-A Li (Phys.), Benjamin Philabaum (Phys.), Sohei Yasuda (Math), Tairan Yuwen (Chemistry)

Others: Neacsu Adrian (Romania), Andrea Altamura (Italy), Manuel Barbero (New York), Haonan Chen (China), Gruian Cornel (IT, Romania), Sandipan Dey (Graduate student, UMBC), Elie Ghosn (Montreal, Quebec), Steven Landy (IUPUI Physics staff), Wei-hsiang Lien (Grad student, National Chiao-Tung Univ., Taiwan), Sorin Rubinstein (TAU faculty, Israel), Craig Schroeder (Grad student, Stanford Univ.), Sahana Vasudevan (8th grade, Miller Middle School, San Jose, CA), Yansong Xu (Brandon, FL), Thierry Zell (Ph.D, Purdue 03)