PROBLEM OF THE WEEK
Solution of Problem No. 5 (Fall 2009 Series)

Problem: Given n points on the sphere of radius 1, show that the sum of the squares of the distances between them does not exceed n^2?
When does this sum equal n^2?

Hint: Use vector algebra.

Solution (by Richard Eden, Graduate student, Math, Purdue University)

For $i = 1, 2, \ldots, n$, let P_i be one of the n points and $\mathbf{r}_i$ the vector from the center of the sphere to P_i. We want to show that $\sum_{i<j} \|\mathbf{P}_i\mathbf{P}_j\|^2 \leq n^2$. Since $\mathbf{P}_i\mathbf{P}_j = \mathbf{r}_j - \mathbf{r}_i$ and $\|\mathbf{r}_i\|^2 = 1$, then

$$\|\mathbf{P}_i\mathbf{P}_j\|^2 = \|\mathbf{r}_i\|^2 + \|\mathbf{r}_j\|^2 - 2\langle \mathbf{r}_i, \mathbf{r}_j \rangle = 2 - 2\langle \mathbf{r}_i, \mathbf{r}_j \rangle.$$

Therefore

$$\begin{aligned}\sum_{i<j} \|\mathbf{P}_i\mathbf{P}_j\|^2 = \frac{1}{2}\sum_{i,j=1}^{n} \|\mathbf{P}_i\mathbf{P}_j\|^2 = \frac{1}{2}\sum_{i,j=1}^{n} \{2 - 2\langle \mathbf{r}_i, \mathbf{r}_j \rangle\} = n^2 - \sum_{i,j=1}^{n} \langle \mathbf{r}_i, \mathbf{r}_j \rangle \\ = n^2 - \left\langle \sum_{i=1}^{n} \mathbf{r}_i, \sum_{j=1}^{n} \mathbf{r}_j \right\rangle = n^2 - \left\| \sum_{i=1}^{n} \mathbf{r}_i \right\|^2 \leq n^2.\end{aligned}$$

Since $\left\| \sum_{i=1}^{n} \mathbf{r}_i \right\|^2 \geq 0$,
we see that the sum is n^2 (i.e. equality occurs) if and only if $\sum_{i=1}^{n} \mathbf{r}_i = 0$.

The problem was also solved by:

Undergraduates: Clara Bennett (Phys), Andy Bohn (Jr. Phys), Artyom Melanich (Fr. Engr.), Kun–Chieh Wang (So. Math), Brent Woodhouse (Fr. Science)

Graduates: Rodrigo Ferraz de Andrade (Math), Vitezslav Kala (Math), Tairan Yuwen (Chemistry)

Others: Neacsu Adrian (Romania), Andrea Altamura (Italy), Manuel Barbero (New York), Gruian Cornel (IT, Romania), Prithwijit De (Kolkata, India), Nathan Faber (CO), Elie Ghosn (Montreal, Quebec), Sleiman Jradi (Freshman, Christopher Newport Univ.), Steven Landy (IUPUI Physics staff), Sorin Rubinstein (TAU faculty, Israel), Tom Sellke (Professor, Purdue)