PROBLEM OF THE WEEK
Solution of Problem No. 8 (Fall 2009 Series)

Problem:

Let a, A be positive numbers. Evaluate

$$\lim_{j\to\infty} \int\limits_0^a \frac{1}{j!} \left[\ln\left(\frac{A}{x}\right)\right]^j dx.$$

Solution (by) Elie Ghosn, Montreal, Quebec

Lets evaluate $I = \int_0^a \frac{1}{j!} [\ln(\frac{A}{x})]^j dx = \lim_{y\to 0} \int_y^a \frac{1}{j!} [\ln(\frac{A}{x})]^j dx$. We have by integration by parts:

$$u = \frac{1}{j!} [\ln(\frac{A}{x})]^j \qquad dv = dx$$
$$du = \frac{-1}{(j-1)!} [\ln(\frac{A}{x})]^{j-1} \frac{dx}{x} \qquad v = x$$

$$\begin{aligned} I_j(y) = \int_y^a \frac{1}{j!} [\ln(\frac{A}{x})]^j dx &= \frac{x}{j!}[\ln(\frac{A}{x})]^j \Big|_y^a + \int_y^a \frac{1}{(j-1)!} [\ln(\frac{A}{x})]^{j-1} dx \\ &= \frac{a[\ln(\frac{A}{a})]^j}{j!} - \frac{y[\ln(\frac{A}{y})]^j}{j!} + I_{j-1}(y). \end{aligned}$$

and by mathematical induction, we deduce:

$$I_j(y) = a\sum_{k=0}^{j} \frac{[\ln(\frac{A}{a})]^k}{k!} - \sum_{k=0}^{j} \frac{y[\ln(\frac{A}{y})]^k}{k!}$$

but $\lim_{y\to 0} y[\ln(\frac{A}{y})]^k = \lim_{y\to 0} y(\ln A - \ln y)^k = 0$ since $\lim_{y\to 0} y(\ln y)^p = 0$ therefore,

$$\int_0^a \frac{[\ln(\frac{A}{x})]^j}{j!} dx = a\sum_{k=0}^{j} \frac{[\ln(\frac{A}{a})]^k}{k!}$$

Finally,

$$\lim_{j\to\infty}\int_0^a \frac{[\ln(\frac{A}{x})]^j}{j!}\,dx = a\sum_{k=0}^{\infty}\frac{[\ln(\frac{A}{a})]^k}{k!}$$
$$= ae^{\ln(\frac{A}{a})} = a\cdot\frac{A}{a} = A$$

The problem was also solved by:

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