PROBLEM OF THE WEEK

Solution of Problem No. 9 (Fall 2009 Series)

Problem: Let, for $n = 0, 1, 2, \ldots$, $f_n(x)$ be defined by the equation $e^x f_n(x) = \sum_{k=1}^{\infty} \frac{k^n x^k}{(k-1)!}$. Show that $f_n(x)$ is a polynomial of degree $n+1$ with integer coefficients.

Solution (by Gabriel Sosa, Purdue University, West Lafayette, IN)

Let's consider the matter of convergence first

$$\lim_{k\to\infty} \frac{\frac{(k+1)^n}{k!}}{\frac{k^n}{(k-1)!}} = \lim_{k\to\infty} \left(\frac{k+1}{k}\right)^n \cdot \frac{1}{k} = \lim_{k\to\infty} \left(1+\frac{1}{k}\right)^n \cdot \frac{1}{k} = 0.$$

So the radius of convergence is ∞.

Now I will use induction. Let $n = 0$. Then

$$e^x \cdot f_0(x) = \sum_{k=1}^{\infty} \frac{x^k}{(k-1)!} = x \cdot \sum_{k=1}^{\infty} \frac{x^{k-1}}{(k-1)!} = x \cdot \sum_{k=0}^{\infty} \frac{x^k}{k!} = x \cdot e^x.$$

So $f_0(x) = x$.

Now assume that for $n = m$, $f_m(x)$ is a polynomial of degree $m+1$ with integer coefficients.

Also notice that $[e^x \cdot f_m(x)]' = \sum_{k=1}^{\infty} \frac{k^{m+1} \cdot x^{k-1}}{(k-1)!}$, and the term by term differentiation is valid for all x. So

$$e^x \cdot f_{m+1}(x) = \sum_{k=1}^{\infty} \frac{k^{m+1} \cdot x^k}{(k-1)!} = x \cdot \sum_{k=1}^{\infty} \frac{k^{m+1} \cdot x^{k-1}}{(k-1)!} = x \cdot (e^x \cdot f_m(x))'.$$

So $e^x \cdot f_{m+1}(x) = x \cdot (e^x \cdot (f_m(x) + f_m'(x)))$. So $f_{m+1}(x) = x \cdot (f_m(x) + f_m'(x))$.

Since $f_m(x)$ has integer coefficients, so does $f_m'(x)$. The degree of $f_m(x)$ is $m+1$, so degree of $f_m'(x)$ is m, and degree of $f_m(x)+f_m'(x)$ is $m+1$. So $f_{m+1}(x) = x\cdot(f_m(x)+f_m'(x))$ is a polynomial of degree $m+2$ with integer coefficients.

The problem was also solved by:

Undergraduates: Kilian Cooley (Fr.), Eric Haengel (So. Math & Physics) Artyom Melanich (Fr. Engr.), Kun–Chieh Wang (Sr. Math), Graduates: Richard Eden (Math), Tairan Yuwen (Chemistry)

Others: Mohamed Alimi (Tunisia), Andrea Altamura (Italy), Manuel Barbero (New York), Brian Bradie (Christopher Newport U. VA), Hongwei Chen (Christopher Newport U. VA), Gruian Cornel (IT, Romania), Tom Engelsman (Chicago, IL), Nathan Faber (CO), Elie Ghosn (Montreal, Quebec), Jeffery Hein (CS & Math, Purdue Univ. Calumet), Steven Landy (IUPUI Physics staff), Jinzhong Li (Shaanxi Normal Univ., China) Wei-hsiang Lien (Grad student, National Chiao-Tung Univ., Taiwan), Angel Plaza (ULPGC, Spain), Sorin Rubinstein (TAU faculty, Israel), Craig Schroeder (Grad student, Stanford Univ.) Steve Spindler (Chicago) Yansong Xu (Brandon, FL), Thierry Zell (Ph.D, Purdue 03)