PROBLEM OF THE WEEK
Solution of Problem No. 10 (Fall 2010 Series)

Problem: Assume that the roots r_1, r_2, r_3 of the polynomial $p(x) = x^3 - 2x^2 + ax + b$ satisfy $0 < r_i < 1$, $i = 1, 2, 3$. Show that

(i) $2 \cdot \sqrt{1 - r_i} \cdot \sqrt{1 - r_j} \le r_k$, (i, j, k) a permutation of 1,2,3;

(ii) $8a + 9b \le 8$;

(iii) the inequality in (ii) is best possible.

Solution (by Steven Landy, IUPUI Physics Dept. Staff)

(i) Since 2 is the sum of the roots, we have $r_3 = (1 - r_1) + (1 - r_2)$ where each bracket is positive. Then the arithmetic–geometric mean theorem says $r_3 \ge 2\sqrt{1 - r_1}\sqrt{1 - r_2}$ and likewise for the other permutations.

(ii) Multiplying the three inequalities from (i)

$$\begin{aligned} r_1 &\ge 2\sqrt{1 - r_3}\sqrt{1 - r_2} \\ r_2 &\ge 2\sqrt{1 - r_1}\sqrt{1 - r_3} \\ r_3 &\ge 2\sqrt{1 - r_1}\sqrt{1 - r_2} \end{aligned}$$

we get

$$r_1 r_2 r_3 \ge 8(1-r_1)(1-r_2)(1-r_3) = 8\Big(1-(r_1+r_2+r_3)+(r_1r_2+r_2r_3+r_1r_3)-r_1r_2r_3\Big).$$

Now using

$$(r_1 + r_2 + r_3) = 2 \qquad (r_1r_2 + r_2r_3 + r_1r_3) = a \qquad -r_1r_2r_3 = b$$

we get

$$-b \ge 8(1 - 2 + a + b) \quad \text{or} \quad 8a + 9b \le 8.$$

(iii) Using $r_1 = r_2 = r_3 = 2/3$ gives $8a + 9b = 8$. So the inequality is the best possible.

The problem was also solved by:

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