PROBLEM OF THE WEEK

Solution of Problem No. 11 (Fall 2010 Series)

Problem: Show that for each real $k \geq 3$ the equation $(\ln x)^k = x$ for $x \geq 1$ has exactly two solutions r_k and s_k where $r_k \to e$ and $s_k \to \infty$, as $k \to \infty$.

Solution: (by Kilian Cooley, Sophomore, Math & AAE)

Since there can be no solution at $x = 1$, we restrict our attention to x strictly greater than 1.

$$(\ln x)^k = x, \quad x > 1$$
$$k(\ln \ln x) = \ln x, \quad x > 1$$

$\ln \ln x = 0$ only when $x = e$, but $(\ln e)^k = e$ is impossible for any real k. Also, if $1 < x < e$, then $(\ln x)^k < 1 < x$, so any solution must occur at $x > e$. Thus we can write

$$k = \frac{\ln x}{\ln \ln x}, \quad k \geq 3, \quad x > e$$
$$x = \exp(\exp(u)), \quad u > 0$$
$$k = \frac{e^u}{u} = g(u)$$
$$\frac{d}{du} g(u) = e^u \frac{u-1}{u^2}.$$

From which we see that $g(u)$ is monotonically increasing for $u > 1$ and monotonically decreasing for $u < 1$, has its only minimum at $u = 1$, and that $\lim_{u \to 0} g(u) = \lim_{u \to \infty} g(u) = \infty$. Since $g(u)$ is continuous it attains every real value $k > g(1) = e < 3$ exactly twice at $v < 1$ and $w > 1$. Also, it is clear that as $k \to \infty$, v and w must tend to 0 and ∞, respectively. Transforming backwards, this corresponds to $x = r_k$ tending to $\exp(\exp(0)) = e$ and to $x = s_k$ tending to ∞.

The problem was also solved by:

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