PROBLEM OF THE WEEK
Solution of Problem No. 13 (Fall 2010 Series)

Problem: Let f be a non–negative, continuous function on the interval $0 \leq x \leq 1$, and suppose that

$$\int_0^x f(t)dt \geq f(x)$$

for all such x.

Solution: (by Craig Schroeder, Ph.D. student, Stanford University)

Let $M = \sup_{[0,\frac{1}{2}]} f(x)$ and $N = \sup_{[\frac{1}{2},1]} f(x)$. Then for $0 \leq x \leq \frac{1}{2}$,

$$f(x) \leq \int_0^x f(t)\,dt \leq \int_0^x M\,dt = Mx \leq \frac{M}{2}.$$

Hence $M = 0$. If on the other hand $\frac{1}{2} \leq x \leq 1$, then

$$f(x) \leq \int_0^x f(t)\,dt \leq \frac{M}{2} + \int_{\frac{1}{2}}^x N\,dt = N\left(x - \frac{1}{2}\right) \leq \frac{N}{2}.$$

Hence $N = 0$. Thus, $f(x) = 0$ for $0 \leq x \leq 1$.

The problem was also solved by:

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