PROBLEM OF THE WEEK
Solution of Problem No. 6 (Fall 2010 Series)

Problem: Show that if f is continuous on $[0,1]$, then

$$\lim_{n\to\infty}\int_0^1 \{nx\}f(x)dx = \frac{1}{2}\int_0^1 f(x)dx.$$

Here, $\{y\} = y - k$ where k is the integer such that $k \le y < k+1$.

Solution (by Elie Ghosn, Montreal, Quebec)

We know that for a continuous function f, the integral $\displaystyle\int_0^1 f(x)dx$ exists and is equal to the limit for $n \to \infty$ of the Riemann Sum $\displaystyle\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right)$. Therefore, we have to show that $\displaystyle\lim_{n\to\infty}\left[\int_0^1 \{nx\}f(x)dx - \frac{1}{2n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right)\right] = 0$. We have for $1 \le k \le n$, k integer,

$$\int_{\frac{k-1}{n}}^{\frac{k}{n}} \{nx\}dx = \int_{\frac{k-1}{n}}^{\frac{k}{n}} (nx - k + 1)dx = \frac{1}{2n}.$$

Therefore,

$$\begin{aligned} S_n &= \int_0^1 \{nx\}f(x)dx - \frac{1}{2n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) = \sum_{k=1}^{n}\left(\int_{\frac{k-1}{n}}^{\frac{k}{n}} \{nx\}f(x)dx - \int_{\frac{k-1}{n}}^{\frac{k}{n}} \{nx\}f\left(\frac{k}{n}\right)dx\right) \\ &= \sum_{k=1}^{n}\int_{\frac{k-1}{n}}^{\frac{k}{n}} \{nx\}\left(f(x) - f\left(\frac{k}{n}\right)\right)dx. \end{aligned}$$

But f, as a continuous function over the compact set $[0,1]$, is also uniformly continuous. Therefore, for a given $\epsilon > 0$ there is $\alpha > 0$ such that:

$$\forall x, y \in [0,1], |x - y| < \alpha \Rightarrow |f(x) - f(y)| < \epsilon.$$

Choosing $n > \dfrac{1}{\alpha}$ gives:

$$\forall x \in \left[\frac{k-1}{n}, \frac{k}{n}\right], \left|f(x) - f\left(\frac{k}{n}\right)\right| < \epsilon.$$

Therefore, for $n > \frac{1}{\alpha}$, $|S_n| \leq \sum_{k=1}^{n} \int_{\frac{k-1}{n}}^{\frac{k}{n}} \{nx\}\epsilon dx = \epsilon\left(\sum_{k=1}^{n} \frac{1}{2n}\right) = \frac{\epsilon}{2}$.

Hence $\lim_{n\to\infty} S_n = 0$.

Also completely or partially solved by:

Undergraduates: Kilian Cooley (So.), Eric Haengel (Jr. Math & Physics), Artyom Melanich (So. Engr.), Yue Pu (Fr. Exchanged student)

Graduates: Shuhao Cao (Math), Richard Eden (Math), Krishnaraj Sambath (Ch.E.), Tairan Yuwen (Chemistry)

Others: Neacsu Adrian (Romania), Manuel Barbero (New York), Gruian Cornel (IT, Romania), Boughanmi Mohamed Hedi (Teacher, Tunisia), Steven Landy (IUPUI Physics staff), Jinzhong Li (Hefei, Anhui, China), Louis Rogliano (Corsica), Sorin Rubinstein (TAU faculty, Israel), Craig Schroeder (Ph.D. student, Stanford Univ.)