

PROBLEM OF THE WEEK
Solution of Problem No. 7 (Fall 2010 Series)

Problem: Let $n \geq 2$ and let $0 < x_i < 1$, $i = 1, 2, \dots, n$. Show that

$$\prod_{i=1}^n (1 - x_i) + \sum_{i=1}^n x_i > 1.$$

Solution (by Lirong Yuan, Freshman, Purdue University)

We can use mathematical induction to solve this problem.

First, we can prove that the inequality is right when $n = 2$

$$\prod_{i=1}^2 (1 - x_i) + \sum_{i=1}^2 x_i = (1 - x_1)(1 - x_2) + x_1 + x_2 = 1 + x_1 x_2 > 1$$

because $0 < x_i < 1$.

Second, suppose the inequality is right when $n = k$ ($k \geq 2$). Thus $\prod_{i=1}^k (1 - x_i) + \sum_{i=1}^k x_i > 1$

$$\text{thus } \prod_{i=1}^k (1 - x_i) > 1 - \sum_{i=1}^k x_i.$$

So

$$\begin{aligned} \prod_{i=1}^{k+1} (1 - x_i) &= \prod_{i=1}^k (1 - x_i) \cdot (1 - x_{k+1}) > \left(1 - \sum_{i=1}^k x_i\right) (1 - x_{k+1}) \\ &= 1 - \sum_{i=1}^k x_i - x_{k+1} + \left(\sum_{i=1}^k x_i\right) \cdot x_{k+1} \\ &= 1 - \sum_{i=1}^{k+1} x_i + \left(\sum_{i=1}^k x_i\right) \cdot x_{k+1} > 1 - \sum_{i=1}^{k+1} x_i. \end{aligned}$$

$$\text{Thus, } \prod_{i=1}^{k+1} (1 - x_i) + \sum_{i=1}^{k+1} x_i > 1.$$

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