PROBLEM OF THE WEEK

Solution of Problem No. 8 (Fall 2010 Series)

Problem: A particle moves on a circle with center O starting from rest at a point P and coming to rest again at a point Q without coming to rest at any intermediate point. Prove that the acceleration vector does not vanish at any point between P and Q, and that it points inward along the radius RO at some point R between P and Q.

Solution (by Jason L. Smith, Professor, Richland Community College, IL)

Without loss of generality, suppose that the circle has radius 1. Also assume that the angular position of the particle as a function of time, $\theta(t)$, is twice–differentiable.

We can write its position from the origin as $\vec{r}(t) = \hat{\imath}\cos(\theta(t)) + \hat{\jmath}\sin[\theta(t)]$. Now take derivatives with respect to time to find the velocity and acceleration.

$$\vec{v}(t) = \Big(-\hat{\imath}\sin(\theta(t)) + \hat{\jmath}\cos[\theta(t)] \Big)\frac{d\theta}{dt}$$

$$\vec{a}(t) = \Big(-\hat{\imath}\cos(\theta(t)) - \hat{\jmath}\sin(\theta(t)) \Big)\left(\frac{d\theta}{dt}\right)^2 + \Big(-\hat{\imath}\sin(\theta(t)) + \vec{\jmath}\cos(\theta(t)) \Big)\frac{d^2\theta}{dt^2}$$

The acceleration can also be written as follows.

$$\vec{a}(t) = -\vec{r}(t)\left(\frac{d\theta}{dt}\right)^2 + \vec{v}(t)\frac{\left(\frac{d^2\theta}{dt^2}\right)}{\left(\frac{d\theta}{dt}\right)}$$

The vectors $\vec{r}(t)$ and $\vec{v}(t)$ are always perpendicular, so $\vec{a}(t)$ will not vanish unless both $\vec{r}(t)\left(\frac{d\theta}{dt}\right)^2$ and $v(t)\frac{\left(\frac{d^2\theta}{dt^2}\right)}{\left(\frac{d\theta}{dt}\right)}$ vanish. By assumption, the object is always moving except for the very beginning and very end of its motion (points P and Q). Therefore $\frac{d\theta}{dt} \neq 0$, so we can conclude that $\vec{a}(t)$ is never zero.

Since $\frac{d\theta}{dt} = 0$ at both endpoints of the interval, its derivative $\frac{d^2\theta}{dt^2}$ must be zero at some interior point t_R in $[t_P, t_Q]$ because of Rolle's Theorem. This implies that

$$\vec{a}(t_R) = -\vec{r}(t_R)\left[\left(\frac{d\theta}{dt}\right)^2\right]_{t_R},$$

which completes the proof.

The problem was also solved by:

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