PROBLEM OF THE WEEK
Solution of Problem No. 1 (Fall 2011 Series)

Problem: Show that $x^{400} + x^{380} + \cdots + x^{20} + 1$ is divisible by $x^{20} + x^{19} + \cdots + x + 1$.

Solution: (by Kilian Cooley, Junior, Math & Aero Engineering, Purdue University)

Let $f(x) = 1 + x + x^2 + \cdots + x^{20}$ and $g(x) = 1 + x^{20} + x^{40} + \cdots + x^{400}$. $f(x)$ divides $g(x)$ if and only if every zero of $f(x)$ is also a zero of $g(x)$, including multiple zeros. f and g are both truncated geometric series, so they can be rewritten as

$$f(x) = \frac{x^{21} - 1}{x - 1}, \quad g(x) = \frac{x^{420} - 1}{x^{20} - 1}$$

From which one sees that the zeros of f and g are precisely those of $x^{21} - 1$ and $x^{420} - 1$ respectively, with the exception in both cases of $x = 1$ where $f(1) = g(1) = 21$. Therefore if $f(r) = 0$, then $r \neq 1$ and, noting that $r^{20} \neq 1$,

$$\begin{aligned} r^{21} - 1 &= 0 \\ r^{21} &= 1 \\ (r^{21})^{20} = r^{420} &= 1^{20} = 1 \\ r^{420} - 1 &= 0 \\ g(r) &= 0 \end{aligned}$$

So any zero of f is also a zero of g. Since the zeros of f are clearly the 21^{st} roots of unity except 1, which are all distinct, every zero of f occurs exactly once in both f and g. Therefore f divides g. Q.E.D.

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