PROBLEM OF THE WEEK

Solution of Problem No. 12 (Fall 2011 Series)

Problem: Let P_i, $1 \leq i \leq n$, be n distinct points in the plane, no three of which are in a straight line. Let $n \geq 3$. Prove that the shortest closed curve which goes through all of these points is a simple polygon, meaning a finite union of edges connected end to end, with each edge sharing each of its endpoints with exactly one other edge, and no other intersections of edges.

Solution: (by Gruian Cornel, Cluj–Napoca, Romania)

There is a finite number $N \leq (n-1)!/2$ of closed curves $C_1, C_2 \ldots, C_N$. For a curve $C = (P_a, P_b, \ldots, P_z, P_a)$, let $l(C) = \|P_aP_b\| + \cdots + \|P_zP_a\|$ and without loss of generality assume that $l(C_1) \leq \cdots \leq l(C_N)$. We prove that the closed curve C_1 has no intersections. Suppose that C_1 has at last a intersection, P_1P_3 and P_2P_4 are edges and they intersect in the point O. We have $\|OP_1\| + \|OP_2\| > \|P_1P_2\|$, $\|OP_2\| + \|OP_3\| > \|P_2P_3\|$, $\|OP_3\| + \|OP_4\| > \|P_3P_4\|$ and $\|OP_1\| + \|OP_4\| > \|P_1P_4\|$. Therefore $\|P_1P_3\| + \|P_2P_4\| > \|P_2P_3\| + \|P_1P_4\|$ and $\|P_1P_3\| + \|P_2P_4\| > \|P_1P_2\| + \|P_3P_4\|$. Corresponding to each point P_k there are exactly two entries, so we have the cases:

1. $C_1 = (P_1, P_3, \ldots, P_2, P_4, \ldots, P_1)$. Eliminate the edges P_1P_3 and P_2P_4, add the edges P_1P_2 and P_3P_4, so we obtain the closed curve $C_1' = (P_1, P_2, \ldots, P_3P_4, \ldots, P_1) \in \{C_2, \ldots, C_N\}$ with $l(C_1) > l(C_1')$, a contradiction.
2. $C_1 = (P_1, P_3, \ldots, P_4, P_2, \ldots, P_1)$. Eliminate the edges P_1P_3 and P_2P_4, add the edges P_2P_3 and P_1P_4, so we obtain the closed curve $C_1' = (P_1, P_4, \ldots, P_3P_2, \ldots, P_1) \in \{C_2, \ldots, C_N\}$ with $l(C_1) > l(C_1')$, a contradiction.

Hence the shortest closed curve has no intersections and it is a simple polygon.

Remark:

1. It is easy to see there can't be closed proper subloops in C_1.
2. The easiest way to show there is some simple polygon through these points may be via the solution of this problem.
3. Some solutions similar to the one above were flawed because the "new" path might not be connected.

The problem was also solved by:

<u>Graduates</u>: Tairan Yuwen (Chemistry)

Others: Charles Burnette (Philadelphia), Hubert Desprez (France), Elie Ghosn (Montreal, Quebec), Steven Landy (IUPUI Physics staff), Kevin Laster (Indianapolis, IN), Sorin Rubinstein (TAU faculty, Israel), Leo Sheck (Faculty, Univ. of Auckland)