PROBLEM OF THE WEEK

Solution of Problem No. 13 (Fall 2011 Series)

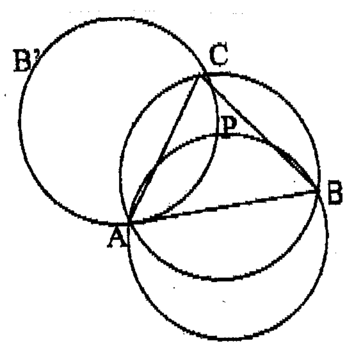
Problem: Prove that the complement of three points in the plane is not a union of discs of radius r, if r is greater than the circumscribed radius (the radius of the circle passing through these 3 points).

Solution: (by Steven Landy, Physics Faculty, IUPUI)

Consider discs of radius R (we'll call these "R discs below") which avoid points A, B, and C. We can certainly cover all points outside the triangle ABC by R discs. Inside the triangle we will show one point cannot be covered. (Picture an r circle with three pins stuck in the circumference. We try to push checkers of radius R against these pins so as to stick into the interior of the r circle and cover the interior points.) If an interior point could be covered by an R disc in any general position, it could be covered by an R disc which also passes through two of the triangle vertices. In addition, if an interior point can be covered by an R disc, it can also be covered by an r disc. Finally if an interior point is on the **circumference** of an r disc passing through the triangle vertices it will not be covered by an R disk passing through the same triangle vertices (and so it can't be covered by any R disc sticking in part ways between those vertices). This is due to the relative convexity of the discs. These statements are easy to show and we will assume them.

Now we will show that if three r discs have the three triangle sides as chords (the checkers are pushed up against the pins), then there is a point P, inside the triangle, which is simultaneously on the circumference of each r disk. This point then cannot be covered by any R disc and the statement will be proved.

Given r circles APC, APB, ABC show circle CPB (not drawn) is an r circle. If we reflect circle ABC thru segment AC, it becomes circle APC, with $B \to B'$. Quadrilateral $APCB'$ is cyclic and $\angle CBA \cong \angle CB'A$. Therefore



$\angle CPA + \angle CBA = \pi$ and similarly

$\angle APB + \angle ACB = \pi$ adding we get

$(\angle CPA + \angle APB) + (\angle CBA + \angle ACB) = 2\pi$

$(2\pi - \angle CPB) + (\pi - \angle CAB) = 2\pi$

$\angle CPB + \angle CAB = \pi$

So circle CPB is circle CAB reflected thru CB. So CPB is an r circle.

So the point P is on the circumference of each r disc and therefore in no R disc.

The proof only applies to acute triangles. There the three r discs meet inside the triangle. For a right triangle or an obtuse triangle the discs meet at the vertices. The proofs for these cases will only be sketched.

In the case of a right triangle, two of the r discs are tangent to one another at the right angle and the third (thru the hypotenuse) hits them both at the right angle vertex. When the disc radius becomes $R > r$, the R disc thru the hypotenuse pulls back from the right angle by a distance dependent on R. So there are points between the "hypotenuse" disc and the right angle which are uncovered.

In the case of an obtuse triangle, the r discs meet at the obtuse angle. When R discs are used, the hypotenuse disc misses the obtuse vertex by a finite distance (which depends on R). There is a neighborhood along the radius from the incenter to the obtuse vertex, near the obtuse vertex which is uncovered by each R disc.

The problem was also solved by:

Graduates: Tairan Yuwen (Chemistry)