PROBLEM OF THE WEEK
Solution of Problem No. 14 (Fall 2011 Series)

**Problem:** Do there exist two different (that is, non–isomorphic) ellipses having the same area and circumference?

**Solution:** (by Tairan Yuwen, Graduate Student, Chemistry, Purdue University)

There do not exist two different (not-isomorphic) ellipses having the same area and circumference. We can prove this by showing that two non-isomorphic ellipses with the same area cannot have the same circumference. Since the area of any ellipse is $\pi ab$ ($a, b$ are lengths of semi-major and semi-minor axes respectively), if we consider a set of ellipses with the same area $\pi A$ and write the length of semi-major axis as $c\sqrt{A}$, then the length of semi-minor axis should be $\sqrt{A}/c$. Since semi-major axis cannot be shorter than the semi-minor axis, there should be $c \geq 1$, besides, each value of $c$ corresponds to a set of ellipses that are isomorphic to each other. Let's place the ellipse in a Cartesian coordinate system with semi-major axis along $x$–axis and semi-minor axis along $y$–axis, then it can be represented as:

$$\frac{x^2}{c^2 A} + \frac{c^2 y^2}{A} = 1$$

and any point on the ellipse has coordinate:

$$x = c\sqrt{A}\cos\theta, \quad y = \frac{\sqrt{A}}{c}\sin\theta \quad (0 \leq \theta < 2\pi).$$

The circumference of the ellipse can be written as the following integral:

$$\begin{aligned}
C &= \int_0^{2\pi} \sqrt{(dx)^2 + (dy)^2} \\
&= \sqrt{A}\int_0^{2\pi} \sqrt{c^2\sin^2\theta + \frac{1}{c^2}\cos^2\theta d\theta} \\
&= 4\sqrt{A}\int_0^{\frac{\pi}{2}} \sqrt{c^2\sin^2\theta + \frac{1}{c^2}\cos^2\theta d\theta} \\
&= 4\sqrt{A}\left(\int_0^{\frac{\pi}{4}} \sqrt{c^2\sin^2\theta + \frac{1}{c^2}\cos^2\theta d\theta} + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sqrt{c^2\sin^2\theta + \frac{1}{c^2}\cos^2\theta d\theta}\right) \\
&= 4\sqrt{A}\int_0^{\frac{\pi}{4}} \left(\sqrt{c^2\sin^2\theta + \frac{1}{c^2}\cos^2\theta} + \sqrt{c^2\cos^2\theta + \frac{1}{c^2}\sin^2\theta}\right)d\theta \\
&= 4\sqrt{A}\int_0^{\frac{\pi}{4}} f(c,\theta)d\theta.
\end{aligned}$$

Now let's just focus on $f(c,\theta)$ and consider its dependence with $c$. Since $f(c,\theta)$ is always positive, we can consider $(f(c,\theta))^2$ instead:

$$
\begin{aligned}
(f(c,\theta))^2 &= \left(\sqrt{c^2\sin^2\theta + \frac{1}{c^2}\cos^2\theta} + \sqrt{c^2\cos^2\theta + \frac{1}{c^2}\sin^2\theta}\right)^2 \\
&= c^2 + \frac{1}{c^2} + 2\sqrt{\left(c^2\sin^2\theta + \frac{1}{c^2}\cos^2\theta\right)\left(c^2\cos^2\theta + \frac{1}{c^2}\sin^2\theta\right)} \\
&= c^2 + \frac{1}{c^2} + 2\sqrt{\left(c^4 + \frac{1}{c^4}\right)\sin^2\theta\cos^2\theta + \sin^4\theta + \cos^4\theta}.
\end{aligned}
$$

Since the function $g(x) = x + 1/x$ is monotone increasing with $x$ as $x \geq 1$, and $x^2, x^4$ are monotone increasing with $x$ as $x \geq 0$, $f(c,\theta)$ is monotone increasing with $c$ as $c \geq 1$ for any fixed value of $\theta$. If we consider the circumference $C$ as a function of $c$, it should be monotone increasing with $c$ as $c \geq 1$, so two ellipses with different values of $c$ must have different circumferences if they have the same area $\pi A$. So there does not exist two non-isomorphic ellipses having the same area and circumference.

**The problem was also solved by:**

Others: Hubert Desprez (Paris, France), Tom Engelsman (Tampa, FL), Steven Landy (Physics Faculty, IUPUI), Peter Montgomery (Microsoft), Craig Schroeder (Postdoc. UCLA)