PROBLEM OF THE WEEK
Solution of Problem No. 2 (Fall 2011 Series)

Problem: Show that $\sin x \geq x - \dfrac{x^2}{\pi}$ if $0 \leq x \leq \pi$.

Solution: (by Thierry Zell, Faculty, Lenoir-Rhyne University)

First, we note that the graphs of $f(x) = \sin x$ and the parabola $g(x) = x - \dfrac{x^2}{\pi}$ both have a vertical symmetry around the axis $x = \dfrac{\pi}{2}$. Thus, it is enough to prove the result for $0 \leq x \leq \dfrac{\pi}{2}$.

The graph of the derivative $g'(x)$ is the line going through the points $(0, 1)$ and $\left(\dfrac{\pi}{2}, 0\right)$. Since the graph of the derivative $f'(x) = \cos x$ is concave on $\left[0, \dfrac{\pi}{2}\right]$ and goes through the same two points, we can conclude that

$$g'(x) \leq f'(x) \quad \text{for all } x \in \left[0, \frac{\pi}{2}\right]. \qquad (1)$$

Since $f(0) = g(0) = 0$, we can write:

$$g(x) = \int_0^x g'(t)dt \quad \text{and} \quad f(x) = \int_0^x f'(t)dt;$$

and the desired inequality follows from integrating Equation (1).

The problem was also solved by:

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