PROBLEM OF THE WEEK
Solution of Problem No. 4 (Fall 2011 Series)

Problem: Show that if

$$u(x) = 1 + \frac{x^3}{3!} + \frac{x^6}{6!} + \frac{x^9}{9!} + \cdots,$$
$$v(x) = x + \frac{x^4}{4!} + \frac{x^7}{7!} + \frac{x^{10}}{10!} + \cdots,$$
$$w(x) = \frac{x^2}{2!} + \frac{x^5}{5!} + \frac{x^8}{8!} + \frac{x^{11}}{11!} + \cdots,$$

then $u^3 + v^3 + w^3 - 3uvw = 1$.

Solution: (by Lirong Yuan, Sophomore, Mathematics, Purdue University)

First, we prove that $(u^3 + v^3 + w^3 - 3uvw)' = 1' = 0$.
Since

$$u'(x) = \frac{x^2}{2!} + \frac{x^5}{5!} + \frac{x^8}{8!} + \frac{x^{11}}{11!} + \cdots = w(x),$$
$$v'(x) = 1 + \frac{x^3}{3!} + \frac{x^6}{6!} + \frac{x^9}{9!} + \cdots = u(x),$$
$$w'(x) = x + \frac{x^4}{4!} + \frac{x^7}{7!} + \frac{x^{10}}{10!} + \cdots = v(x),$$

hence

$$\begin{aligned}(u^3 + v^3 + w^3 - 3uvw)' &= 3u^2u' + 3v^2v' + 3w^2w' \\ - 3[(uv)'w + uvw'] &= 3u^2u' + 3v^2v' + 3w^2w' \\ - 3[(u'v + uv')w + uvw'] &= 3u^2w + 3v^2u + 3w^2v \\ - 3[(wv + u^2)w + uv^2] &= 3(u^2w + v^2u + w^2v) \\ - 3(w^2v + u^2w + uv^2) &= 0.\end{aligned}$$

Next, we prove that $u^3 + v^3 + w^3 - 3uvw = 1$.
Since $(u^3 + v^3 + w^3 - 3uvw)' = 0$,
hence

$$\int (u^3 + v^3 + w^3 - 3uvw)' dx = \int 0 \, dx$$
$$u^3 + v^3 + w^3 - 3uvw = c.$$

Since $u(0)=1, v(0)=0, w(0)=0$,
hence

$$c=u^3+v^3+w^3-3uvw=1+0+0-0=1.$$

As a conclusion, $u^3+v^3+w^3-3uvw=1$.

The problem was also solved by:

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