PROBLEM OF THE WEEK
Solution of Problem No. 5 (Fall 2011 Series)

Problem: A coast artillery gun can fire at any angle of elevation between 0° and 90° in a fixed vertical plane. If muzzle velocity is constant ($= v_0$), determine the set H of points in the plane (and above the horizontal) which can be hit. (Neglect air resistance.)

Solution: (by Bennett Marsh, Freshman, Engineering, Purdue University)

Parametric equations for the flight of the projectile, based on simple kinematic equations from physics (and assuming the gun is at the origin, with $g =$ gravitational acceleration of Earth), can be written as

$$y(t,\theta) = -\frac{1}{2}gt^2 + v_0 t \sin\theta$$
$$x(t,\theta) = v_0 t \cos\theta.$$

Solving for t in terms of x and substituting into the y equation, we get

$$y(x,\theta) = \left(-\frac{g}{2{v_0}^2}\sec^2\theta \right)x^2 + x\tan\theta.$$

For any given x value, we must find the value for θ that maximizes y to get an upper bound on the points that the gun can reach. In other words, we need to find the θ such that

$$\frac{\partial y}{\partial \theta} = \sec^2\theta\left(\frac{-gx^2}{{v_0}^2}\tan\theta + x \right) = 0.$$

Since $\sec\theta$ is never 0, to solve the equation we must set the second part equal to zero, and we find that

$$\theta = \tan^{-1}\frac{{v_0}^2}{gx}.$$

Plugging this value in for θ in the equation for y, we get,

$$y_{\max}(x) = \frac{{v_0}^2}{2g} - \frac{gx^2}{2{v_0}^2}.$$

This gives us an upper bound for the height of the projectile at any given x value. So the set of points H that the gun can reach lies in between the zeros of $y_{\max}$ and below the parabola itself, or the set of all points satisfying

$$-\frac{{v_0}^2}{g} \le x \le \frac{{v_0}^2}{2g} \quad \text{and} \quad 0 \le y \le \frac{{v_0}^2}{2g} - \frac{gx^2}{2{v_0}^2}.$$

The problem was also solved by:

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