PROBLEM OF THE WEEK Solution of Problem No. 6 (Fall 2011 Series)

Problem: Given nine lattice points in space, show that there is an interior lattice point on at least one segment joining a pair of them.

Note: A "lattice point" is a point whose x-, y-, and z-coordinates are all integers.

Solution: (by Kaibo Gong, Senior, Mathematics, Purdue University)

Consider function $\phi(a_1, a_2, a_3) = (a_1 \mod 2, a_2 \mod 2, a_3 \mod 2) \ (a_1, a_2, a_3 \in \mathbb{Z})$. Thus this function has 8 results.(0, 0, 0)(0, 0, 1)(0, 1, 0)(0, 1, 1)(1, 0, 0)(1, 0, 1)(1, 1, 0) and (1, 1, 1). (Here 0 means a_i is even and 1 means odd.)

Thus for the 9 lattice points in space from pigeon-hole thm, at least two points will get the same result. Say, $\psi(m) = \psi(n)$ $m = (m_1, m_2, m_3)$ $n = (n_1, n_2, n_3)m_i \in \mathbb{Z}, n_i \in \mathbb{Z}$. Thus $m_1 \equiv n_1 \mod 2$ $m_2 \equiv n_2 \mod 2$ $m_3 \equiv n_3 \mod 2$ which means $m_1 - n_1 = 2k_1$, $m_2 - n_2 = 2k_2$, $m_3 - n_3 = 2k_3$, $k_1, k_2, k_3 \in \mathbb{Z}$. Thus, the point $k = (k_1, k_2, k_3)$ is the mid-point of the line segment mn.

Thus k, the mid-point of mn is a lattice point.

The problem was also solved by:

<u>Undergraduates</u>: Seongjun Choi (Jr. Math), Kilian Cooley (So.), Yixin Wang (So. ECE), Lirong Yuan (So.), Xinghang Yuan (ME)

<u>Graduates</u>: Vaibhav Gupta (ECE), Biswajit Ray (ECE), Tairan Yuwen (Chemistry), Samson Zhou (CS), Guangwei Zhu (ECE)

<u>Others</u>: Manuel Barbero (New York), Max Clark, Gruian Cornel (IT, Romania), Hubert Desprez (Jussieu University, France), Tom Engelsman (Tampa, FL), Elie Ghosn (Montreal, Quebec), Jae Woo Jeon (Seoul, Korea), Brendan Kinnell (Richmond, VA), Martin Kleinsteuber (Germany), Steven Landy (IUPUI Physics staff), Kevin Laster (Indianapolis, IN), Matt Mistele (FL), Achim Roth (Data Protection Officer, Germany), Sorin Rubinstein (TAU faculty, Israel), Craig Schroeder (Postdoc. UCLA), Leo Sheck (Faculty, Univ. of Auckland), Steve Spindler (Chicago), Thierry Zell (Faculty at Lenoir–Rhyne University)