PROBLEM OF THE WEEK

Solution of Problem No. 9 (Fall 2011 Series)

Problem: Rearrange the series $* = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + (-1)^{k+1} \cdot \frac{1}{k} + \cdots$ so that the first term of the rearranged series is a positive term of $*$, the next two terms are negative terms of $*$, the next three terms are positive terms of $*$, etc., and that the positive terms are decreasing and the negative terms increasing. So the rearranged series is

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} - \frac{1}{6} - \frac{1}{8} - \frac{1}{10} - \frac{1}{12} + \frac{1}{9} + \cdots .$$

Is the rearranged series convergent?

Solution: (This solution is a conflation of several of the solvers' solutions.)

Let S_n be the nth partial sum of the original series S and T_n be the nth partial sum of the rearranged series T. Let q_n stand for $\Sigma_{k=1}^{n} k = n(n+1)/2$. It is easily seen that if n is odd T_{q_n} is a sum of more of the positive terms than negative terms of the original series, while if n is even there are more negative terms. This implies that there exists k_n between q_n and q_{n+1} such that T_{k_n} is composed of equal numbers of positive and negative terms. Together with the fact that the positive terms appear as summands in T in the order they appeared in S, as do the negative terms, we see $S_{k_n} = T_{k_n}$, and so T_{k_n} converges to the sum S. Since $T_k, q_n \le k \le q_{n+1}$ is monotone, to prove convergence it now suffices to show that $|T_{q_{n+1}} - T_{q_n}|$ approaches zero as n approaches infinity. Now $T_{q_{n+1}} - T_{q_n}$ is the sum of $n+1$ terms of S, each of them smaller in absolute value than the k_{n-1}st term of S which is itself smaller in absolute value than the absolute value of the q_{n-1}st term of S, which equals $\frac{1}{q_{n-1}}$. Thus $|T_{q_{n+1}} - T_{q_n}| \le (n+1)/q_{n-1}$ which $\to 0$ as $n \to \infty$.

The problem was also solved by:

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